VIBRATIONS AND STABILITY OF ELASTICALLY RESTRAINED EXPANSION BELLOWS IN PIPELINES

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CERTIFICATE

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ABSTRACT

The present study is to theoretically investigate the effect of elastically restrained ends on the axial and transverse vibrations of expansion bellows. The work carried out is relevant, as the end conditions of bellows in general at industrial sites are quite complex giving rise to unequal rotations at the ends. Physically, the system may be represented as a short pipe nipple welded at both ends of bellows and then the flanges welded to the other end of pipe nipple

The bellows are modeled as an equivalent beam including rotatory inertia as has been considered by earlier researchers and EJMA. The general method followed in deriving exact frequency expressions in the present work is based on "Separation of Variables" approach in order to find the axial and transverse natural frequencies of single bellows. This method is nearly well adopted in analyzing the vibration of Universal (double) bellows also.

Surprisingly, the previous works contributed by researchers Li Ting-Xin et al and Jakubauskas V.F et al. [4,5,15], show that they had paid more attention to study the effect of classical fixed-fixed type of boundary conditions only and obtained axial and transverse natural frequencies. It is also seen that no provision has been made to these effects in EJMA code also while computing the axial and lateral vibration frequencies for single and Universal bellows and so may not be treated as accurate estimates.

The influence and variation of the rotational elastic restraint parameter is studied. It is seen that the effect of this parameter on natural frequencies is quite significant. The effect of variation of the internal pressure and flow velocity of fluid in the bellows is also studied. The results obtained thereon by solving the closed form frequency equations are then confirmed numerically by using the finite element method and are found to be in below 10% of error for all the cases.

The particular case of bellows with fixed-fixed boundary condition is a reduced case wherein the elastic restraints against rotation are compared well with the experimental, numerical and theoretical results for both axial and transverse vibrations respectively. All the results are thereon plotted for variation of rotational stiffness internal pressure and flow velocity versus axial and transverse frequencies.

LIST OF PAPERS CONTRIBUTED

1. Radhakrishna. M & Kameswara Rao. C. “Axial Vibrations of U Shaped Bellows with Elastically Restrained End Conditions”, Paper communicated to Journal of Thin Walled Structures, Elsevier Applied Science Publishers, Great Britain.

2. Radhakrishna. M & Kameswara Rao. C. “ Finite Element Analysis of Axial Vibrations of Single Bellows Restrained Against Rotation”, Proceedings of Second International Conference on Vibration Engineering and Technology of Machinery VETOMAC-2, 16-18 Dec 2002, Mumbai, India.

3. Radhakrishna. M & Kameswara Rao. C. “Transverse Vibrations of Single Bellows Expansion Joint Restrained Against Rotation” Tenth International Conference on Nuclear Engineering, USA, ASME Paper # 22090, April 14-18, 2002.

4. Radhakrishna. M & Kameswara Rao. C. “Finite Element Analysis of Transverse Vibrations of Single Bellows Expansion Joint Restrained Against Rotation”, Paper communicated to Journal of Thin Walled Structures, Elsevier Applied Science Publishers, Great Britain

5. Radhakrishna. M & Kameswara Rao. C. “Transverse Vibrations of Double Bellows Expansion Joint Restrained Against Rotation” Tenth International Conference on Nuclear Engineering, USA, ASME Paper # 22092, April 14-18, 2002.

6. Radhakrishna. M & Kameswara Rao. C. “Seismic Response of Single Bellows Restrained Against Rotation” Eleventh International Conference on Nuclear Engineering, Japan, ASME Paper # 36400, April 20-24, 2003.

7. Radhakrishna. M & Kameswara Rao. C. “Vibrations of Fluid Filled Bellows –A-State-of-the-Art”, Proceedings of the National Symposium on Advances in Structural Dynamics and Design, Jan 9-11, 2001.

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Nomenclature

A_c : cross sectional metal area of one bellow convolution, m^2

A_o : Bellows effective area corresponding to mean diameter of bellow convolution, m^2

C_d, C_f, C_p : factor used in design calculations to relate U shaped bellows convolution segment

C_m : Material strength factor

C_t : temperature correction factor for bellows fatigue life below creep range

C_{wb} : longitudinal weld joint efficiency factor

C_z : transition point factor

D_b : Inside diameter of cylindrical tangent and bellows convolution, m

D_m : Mean diameter of bellows convolution, m

E_b : Modulus of elasticity of bellows material at design temperature, N/m^2

E_{br} : Modulus of elasticity of bellows material at room temperature, N/m^2

e : Total equivalent axial movement per convolution, mm

f_{iu} : Bellows theoretical axial elastic spring rate per convolution, N/m

f : frequency, Hz

h : convolution depth less bellows material thickness, in

J *: mass moment of inertia per unit length, kg/m*

k : factor which considers the stiffening effect of attachment weld and the end convolution on pressure capacity of bellows tangent

L_b : bellows convoluted length, m

L_C	: bellows tangent collar length, m
L	: bellows tangent length, m
m_{tot}	: *total mass of bellows per unit length includes mass of bellows and fluid mass*
M_s	: *Support mass of supports, kg*
$N_{1,2\ldots}$	**: mode number**
n	: number of plies of thickness t
N	: number of convolutions
N_C	: fatigue life, number of cycles to failure, cycles
P	: internal pressure force, MPa
q	: convolution pitch, mm
R_1 & R_2	: Rotational Stiffness
R_m	: mean radius of bellows, m
S_{ab}	: allowable bellows material stress at design temperature, N/m^2
S_{ac}	: allowable collar material stress at design temperature, N/m^2
S_t	: total stress, N/m^2
S_y	: yield stress at design temperature, N/m^2
t	: bellows thickness of one ply, mm
t_p	: bellows material thickness for one ply, corrected for thinning, mm
EI	: bending stiffness, Nm^2
t	: time
T	: Rotational non-dimensional parameter (T_1, T_2)
v	: velocity of flow, m/s

w, y	: deflection, mm
x, z	: axial coordinate
ρ_b	: mass density of bellows material, kg/cu.m.
ρ_f	: mass density of fluid flowing through bellows, kg/cu.m.
ω	: Frequency in radians/s

Note: Variables have also been defined wherever required in chapters

CHAPTER 1

INTRODUCTION

1.1 Preamble

Generally, U-shaped bellows expansion joints as shown in Fig 1.1, are used as flexible elements in piping systems to absorb the relative movements resulting from thermal expansion in pipe system. In order to prevent excessive stress and avoid any damage to either the equipment or pipeline expansion bellows are employed. However, it is seen that bellows fail to give many years of satisfactory performance when they are not properly designed for specified piping system conditions.

They are used in chemical plants, nuclear power systems, heating and cooling systems and cryogenic plants. Typical service conditions could be as severe as pressures ranging from full vacuum to 1000psig and temperatures from minus 420deg F to 1800deg F. Therefore the design and fabrication of an expansion bellow requires serious attention as it falls into the category of a highly engineered product.

Failures of bellows can occur because of many reasons, but experience has shown that bellows failure due to mechanical or flow induced vibration falls into fairly distinct category. Hence, the design of bellows for vibrations has to be viewed seriously. For example, the reciprocating machines such as compressors are usually connected to piping system through flexible elements like expansion bellows in order to prevent noise and stress propagation over the whole piping system. The bellows in

such piping systems are subjected to severe vibrations. In piping systems designed for the thermal and nuclear power plants located in earthquake prone regions, the expansion bellows are to be designed to take care of the seismic excitations also. Non-availability of sufficient space to route the piping is commonly envisaged in a chemical plant. Expansion bellows employed in such piping systems are subjected to axial movement, angular rotation, lateral deflection or any combination of these, giving rise to forces and moments. The influence of such forces and moments shall be considered in determining the axial and transverse natural frequencies of bellows while carrying out dynamic analysis.

The most important part of the expansion bellow is the convoluted portion. The bellows convolutions are available in variety of shapes, the most popular of which consists of flat circular rings connected to two toroidal half rings, forming the root and tip of the convolution. These are called as U-convolutions. This shape is most widely used because it permits formation of complete bellows either mechanically or hydraulically from a single piece of cylinder and this thesis is restricted to the study of such U-shaped bellows only.

1.2 The Bellows

The bellows are the convolutions and flexible portion of the expansion joint. They must be strong enough circumferentially to withstand the pressure and flexible enough longitudinally to accept the deflections for which they are designed. Hence,

the strength with flexibility is a unique design problem that is not often found in other components in industrial equipment.

Most of the structures are designed to inhibit deflection when acted upon by outside forces. Since the bellows must accept deflections repetitively, the deflections will give rise to stresses. These stresses must be kept as low as possible so that the repeated deflections will not result in premature fatigue failures. Reduction of bending stress for a given deflection is easily achieved by simply reducing the thickness of convolution. However, in order to withstand the pressure, the convolution must have a thickness great enough that the pressure induced membrane stress is equal to or less than the allowable stress level of the material chosen at the design temperature. This conflicting need, thickness for pressure and thinness for flexibility is the unique design problem faced by the designer.

Bellows are normally categorized as - formed membrane (single-ply un-reinforced / reinforced & multi-ply) and formed head (only flanged or flanged-flued). This kind of categorization is based on the type of numerical method used and generally applicable for an elastic and small deflection analysis. Single-ply un-reinforced type bellows are generally evaluated using linear thin shell theory. On the other hand, exact analysis of a multi-ply bellow or a ring-reinforced type of bellow requires consideration of non-linear interaction between the plies, or between the ring and plies respectively. The flanged and flued type of expansion joint is characterized by large wall thickness and is governed by thick wall equation [10].

A summary of the major differences between various types of bellows is illustrated in Table 1.1 [5].

Table 1.1 Differences between bellow configurations

Type of Bellow	Linear thin shell theory	Thick wall behavior	Nonlinear interaction	Sharp Corners	Typical use
U-shaped	Yes	No	No	No	Most common
Semi-toroidal	Yes	No	No	No	High pressure
S-shaped	Yes	No	No	No	As U-shaped
Omega	Yes	No	No	No	High pressure
Multi-ply	No	No	Yes	No	Low stiffness
Reinforced	No	No	Yes	No	High pressure
Flanged & flued	No	Yes	No	No	Exchanger shell

1.3 Formed Membrane

The formed membrane type is popularly known as the bellows type. It is made by forming convolutions from a thin (a few thousands of an inch) shell of a corrosion resistant material, such as austenitic steel or a nickel base alloy. Figures 1.1 and 1.2 show typical views of solid model and cross-section of the un-reinforced that are used in low-pressure applications.

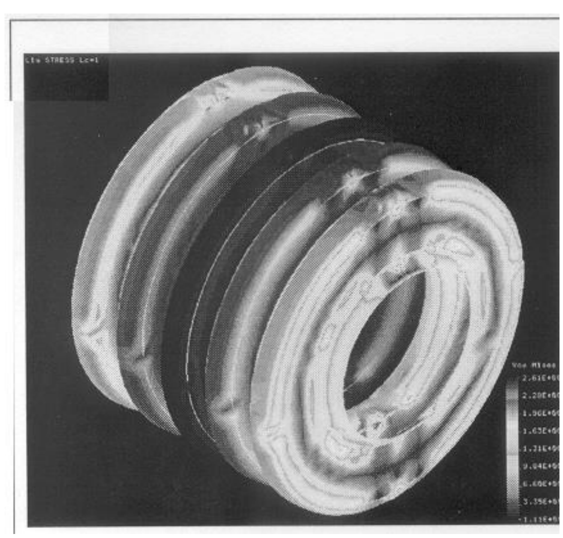


Fig 1.1 Solid Model View of Bellows

Reinforcing rings are added when instability or *Squirm* of the bellows is of concern. All bellows have a critical pressure at which they become unstable. Instability occurs in two modes – column instability (or squirm), and in-plane deformation of the convolution side wall. Squirm is the phenomena whereby the centerline of a straight bellows develops a sideways or lateral bow.

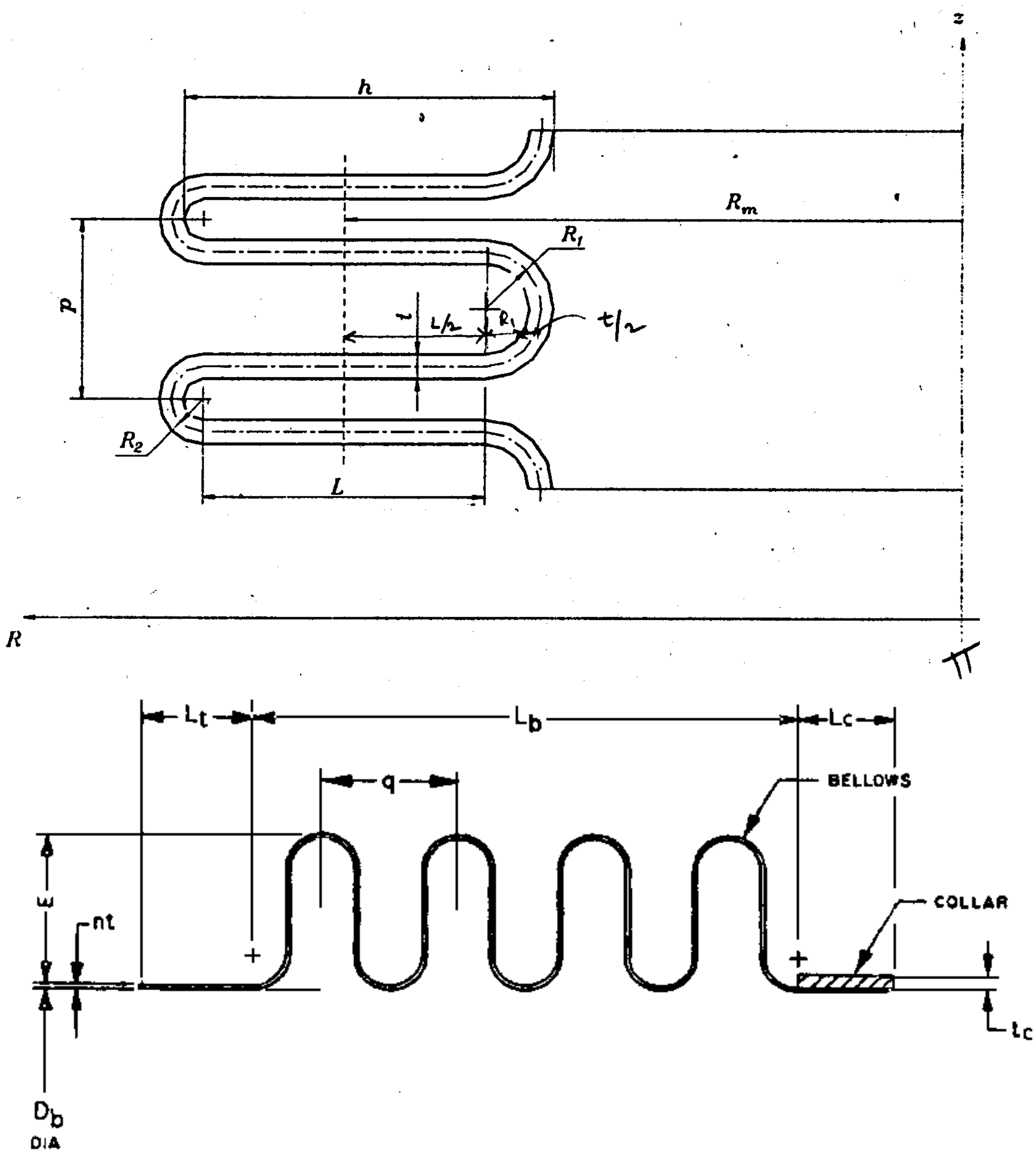


Fig 1.2 Cross-sectional views of Bellows

1.4 Basis of Design and Theory

Differential longitudinal expansion between the shell and the tube in a heat exchanger or in a pipeline is a problem that occurs either because of temperature or pressure. The effect of pressure induced differential growth is often over-looked in the design of fixed tube sheet heat exchangers. Physically, it means that there is a mismatch in the axial deformation of shell and tube bundle that is caused by the difference in the state of their pressure loading.

1.5 Type of Deflections

In order to properly apply expansion joints to piping systems it is necessary to visualize the deflections that result from thermal expansion or movements and vibrations of equipment and structures. All expansion joints do not accept the same types of deflection. Many can accept certain loads and moments, while others are incapable of resisting externally applied forces. Hence, understanding the type, magnitude and direction of forces and deflections is critical from safety point of view.

1.5.1 Axial Deflection

Axial refers to being parallel to the centerline of the expansion joint. COMPRESSION is the axial deflection that will shorten the bellows length. Often, thermal expansion in the piping will cause the expansion joint to be compressed. EXTENSION is the axial deflection that stretches the expansion joint. Piping that operates at temperatures lower than ambient such as in cryogenic systems –will contract, causing the expansion joint to stretch as shown in Fig 1.3.

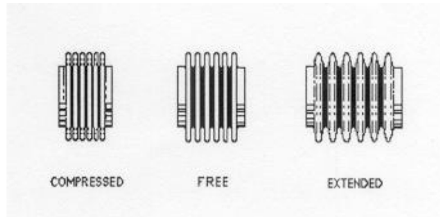


Fig 1.3 Axial deflection

1.5.2 Lateral Deflection

Lateral or Transverse refers to the direction perpendicular to the centerline of the expansion joint and shown in Fig 1.4. This movement occurs with both the ends of the expansion joint remaining parallel to each other, with their centerlines being displaced. It is not uncommon to find different lateral deflections can occur in more than one plane. As the expansion joint is circular in cross-section, all the various deflections must be resolved into a single resultant lateral deflection –which is square root of the sum of the squares of the individual deflections.

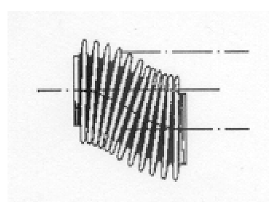


Fig 1.4 Lateral deflection

1.5.3 Angular Deflection

When an expansion joint experiences bending about its center, which is on the centerline and halfway between the ends of the bellows, the deflection is referred to as angular deflection and is shown in Fig 1.5. It can occur in any plane that passes through the centerline and is the maximum of the various deflections, and not the vector sum as in the lateral case.

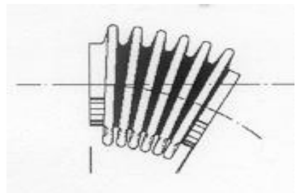


Fig 1.5 Angular deflection

1.5.4 Torsional Deflection

Figure 1.6 depicts torsion deflection of bellows. Torsion refers to twisting one end of the bellows with respect to the other end, about the bellows centerline. The expansion joints are normally not expected to accept torsion deflection.

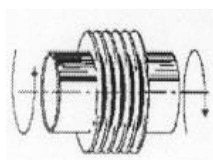


Fig 1.6 Torsional deflection

1.5.5 Cyclic Deflections and Cyclic Life

Most deflections are repeated a number of times during the life of the piping system, since the deflections usually are produced by change in temperature that occur each time the system is started and stopped. Each time a deflection occurs it is a cycle. The number of cycles is important to assure the proper design of an expansion joint, since each design has a finite predictable life.

Vibrations cause repetitive deflections and can cause a premature failure of expansion joint. Even though these deflections are small in magnitude, they usually accumulate huge number of cycles in a short period of time. Since bellows are

metallic structures, they have a specified resonant natural frequency and when driven by outside vibrations of same frequencies, they can magnify the deflections until they exceed the yield strength of the bellows material and induce early fatigue failure. For example, when a piping system is known to have equipment that can produce vibrations such as pumps, compressors and other motor or turbine driven devices whose rotational speeds or frequencies are of importance.

1.6 Types of Expansion Joints

There are different types of expansion joints – each has a definite advantage and limitation and a unique design style of its own. The thesis cover only two common types of expansion joints that are as follows –

1.6.1 Single Bellows Expansion Joint

The single bellows expansion joint as shown in Fig 1.7, has a simple bellows element with end connections. It will deflect in any direction or plane, but requires the piping to be controlled to the direction of the movement. This expansion joint will not resist any deflections with any force other than the resistance of bellows, which is a function of the spring rate and deflection.

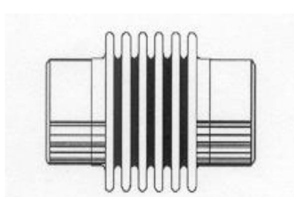


Fig 1.7 Single Bellows

1.6.2 Universal OR Double Bellows Expansion Joint

The Universal expansion joint as shown in Fig 1.8, consists of two bellows separated by a pipe section or spool. The primary purpose of this kind of an arrangement is to have a unit that will accept large amounts of lateral deflection. The amount of lateral deflection that it can accept is a function of the amount of angulation of each bellows can absorb and the distance between the bellows. So, for a given bellows element, the amount of lateral deflection capability can be increased or decreased by simply changing the length of the center spool.

The design analysis of an expansion bellow is quite complex because it is stressed due to operating pressure, axial deflection and at the same time has to

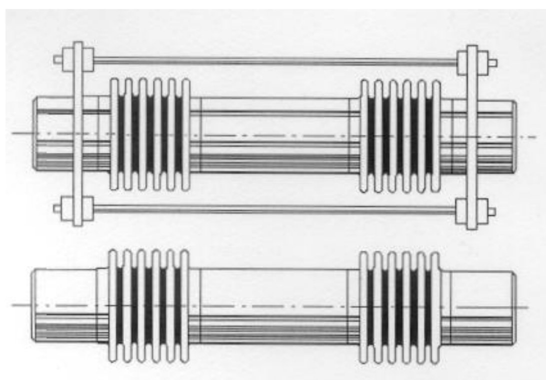


Fig 1.8 Double Bellows or Universal Expansion Joint

remain stable and work up to the designed fatigue life. Any assumed bellow geometry which is defined by diameter, depth of convolution, pitch, number of convolutions, thickness and number of plies for a given material of construction and manufacturing technique employed etc., has to satisfy all the above mentioned criteria. All these factors are interdependent and change in any one of the variables affect the overall bellow design.

1.7 Literature Survey and Review of Vibrations of Fluid Filled Bellows

Literature survey was carried out to review the present state-of-the-art in the area of design of metallic bellows and fluid-filled bellow vibrations. The aim is to identify the direction required for further theoretical investigations to be carried out, in order to present a fairly good and acceptable theoretical model for predicting the dynamic behavior of fluid- filled bellows.

A thorough survey of literature reveals that most of the early investigators on bellows have studied at great depth the static response of bellows - stress patterns, design aspects of bellows and the bellow failure modes like root-bulge or collapse, evaluation of creep rupture and creep-fatigue and stability analysis. Quite often bellows are also used to arrest the propagation of vibrations in the piping systems. Being very flexible, bellows are susceptible to vibrations and more so can be easily excited by internal flow.

It was only in the early 1990, when some sporadic failures of bellows were reported due to high flow velocities of internal fluid - the failure of bellows due to Flow Induced Vibrations (FIV) came to the fore. The significance of dynamic response on bellows reliability did not attract immediate attention, until bellow failures became a common place particularly in power industry. The last decade has therefore witnessed a vigorous research activity in the field of flow induced vibration of bellows – to understand the phenomena [2]. The goal herein is to present the early developments on the subject and to attempt a physical insight into the vibration

problem and summarize the various methods proposed in the literature- theoretical and numerical and present further direction on flow induced vibrations in bellows.

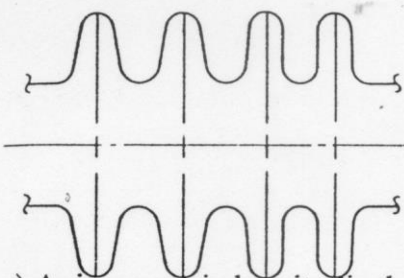
1.7.1 General Information about Flow Induced Vibrations

Flow induced vibrations (FIV) of elastic bodies involve the interaction between the elastic and inertia forces of the elastic body and the fluid. It is seen that the vibration excitation mechanisms are not well understood, since the FIV phenomena is complex and diverse. The exact determination of the nature of the interaction between the structure and fluid and the magnitude of force of the interaction is extremely difficult. The FIV appears in tubes of heat exchangers and steam generators –investigated by Paidoussis (1979), Shin and Wambsgnass (1975); in nuclear fuel assemblies, Oldaker et al. (1973); in hydro technical components and structures, Weaver (1976); in naval and space industry, Sainsbury and King (1971). The wind excitation of buildings and aboveground structures is another area where FIV is applied [13].

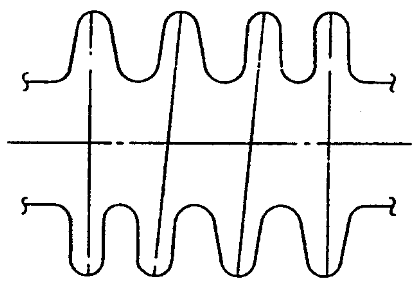
The list of published works in FIV is increasing rapidly because of the diversity of the subject. Naudascher (1967), Toebes (1965) and others attempted to classify the flow induce vibrations. However, according to Weaver (1989) FIV is characterized as follows-

- Forced vibrations
- Self-controlled vibrations
- Self-excited vibrations

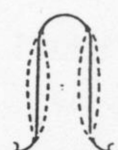
Figures 1.9 {(a), (b), (c) & (d)} represent the various modes of vibration of bellows.



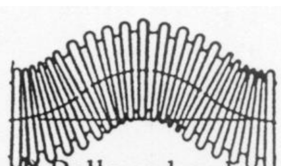
a) Axisymmetric longitudinal mode



b) Nonsymmetric longitudinal mode



c) Local convolute bending mode



d) Bellows bending mode

Forced vibrations arise because of various sources. Some of them include the vibrations due to turbulent flow, vibrations as a response of tall buildings or aircraft structures to wind gusts, vibrations of ship propeller blades excited due to the periodical flow near the ship hull, and the response of a pipe carrying fluid. It is believed that whether the excitation phenomena are random or periodic, the motion of the structure has no feed back effect on the fluid forces. Therefore the excitation force can be studied separately from the vibration problem using a rigid model. So this separation of the whole response problem into two independent ones greatly simplifies the solution.

In self-excited vibration problems some periodicity in the flow exists in case of stationary structure. When this periodicity coincides with the natural frequency of the structure resonance takes place. The vibration amplitude increases until the structural motion starts to control the fluid excitation force. Under, these conditions in some fluid velocity range, the vibration response of the system is controlled not by fluid velocity, but by vibrating structure. The fluid velocity range is the lock-in region. A common example of periodicity in the flow is the vortex shedding, vibrations of stacks /towers or turbine blade. Such vibrations can be controlled or prevented by changing the stiffness of the structure in order to alter the natural frequency or more effectively to change the geometry of structure to alter the fluid excitation.

Self-excited vibrations appear in such systems when the motion of the structure itself creates the periodic fluid force that in turn amplifies the vibration of structure. The periodic force doesn't exist in the absence of structural motion, as in case of self-controlled vibrations and this is the main distinction between the two types. Examples of self-excited vibrations are bending-torsion vibrations of aircraft wings, the oscillation of gate seals and the vibrations of vertical lift gates. Because of the interaction of fluid and elastic forces, both self-controlled and self-excited vibrations are called fluid-elastic vibrations.

In FIV of liquid flows, it is important to take into account the inertia of the fluid because this is usually great enough to change the vibration frequencies considerably and the modes of vibration of the system. The motion of any rigid or elastic body in a fluid is accompanied by flow of fluid around the body. Considering the fluid to be perfect and incompressible and its flow as steady and irrotational, both the drag and lift forces for the body possess symmetry in center according to D' Alembert and equal to zero. However, if the motion is not uniform, the flow is not steady. In this case the flow generates a drag force on the body and this force appears as if the inertia of the body has been increased. Since the vibration motion is non-uniform, the mass of the vibrating body appears to have increased by an "Additional mass". This phenomenon is significant because it can lead to a drop in the natural frequency of the vibrating body, if the fluid density is sufficiently high. The first attempt to evaluate theoretically the additional mass concept in bellows was made by Gerlach (1969).

1.7.2 Early Developments

❖ Theoretical Methods

Love [3] was the first to propose the theory of elastic shells – which resulted in two formulations for axi-symmetric shells- Love-Meissner and E.Meissner formulations. The earliest stress/deflection analysis of bellows relied mainly on simple beam [11] or beam and cylinder approximations. It is observed that these gave only qualitative but not quantitative assessments of bellows stress/deflection. The analytical solutions were based on energy methods and asymptotic integration – considered only axial loading. However, a number of investigators later used energy methods to evaluate the response of bellows to axial force / deflection. Salzman evaluated the pressure stresses of a U-shaped bellow using a three-term series to represent the meridoinal bending deformation. Dahl [9] used a four-term series for an Omega shaped bellows. Turner and Ford [44] used a five-term series to represent circumferential stress and evaluate semi-toroidal bellows. Turner extended this work to U-shaped bellows – evaluated the pressure loading by approximating the bellows to beam theory. Laupa and Weil [39] evaluated U-shaped and semi-toroidal bellows using a five term series and expressed the bending deformation of a toroid and a circular plate to represent the convolution sidewall. They repeated the same work using shell theory. Clark [8] developed solutions for axial deflection of Omega type bellows using asymptotic integration.

Anderson [1,2] used the asymptotic solution given by Clark, to develop solutions for deflection stresses in U-shaped bellows. He also introduced corrections

to compensate for the approximations suggested by Clark. Pressure stress was also evaluated using the beam theory and correction factors were introduced as charts in order to relate the simplified equations to the shell behavior. This work by Anderson later formed the basis for the stress equations for un-reinforced / reinforced bellows in *the Expansion Joint Manufacturers Association Handbook* [EJMA 10]. The state-of-the-art in bellow analysis was clearly presented by Becht IV [5] and given in Table 1.2.

Table1.2 State-of-the-art in Bellow Analysis

Type of Bellow	Elastic	Elastic/Plastic	Creep	External pressure	Internal Pressure
U-shaped	α	α	α	α	☒
Multi-ply	□	□	□	☒	☒
Reinforced	☒	□	□	☒	☒
Flanged & flued	☒	□	□	☒	☒

☒ **Simplified methods of evaluation are available**
α **Rigorous numerical analysis performed**
□ **Has not been evaluated**

❖ **Numerical Methods**

Numerical methods gained importance only with the advent of digital computation. These methods are the *finite difference, numerical integration and finite element.* Krauss [35] solved the shell equations using the first two numerical techniques. The finite element method involves partitioning of the shell into finite elements to get more exact approximations.

❖ **Finite Difference**

In this method the second order differential equations be replaced by a set of linear algebraic difference equations. Their derivatives are written over a finite length instead of infinitesimal length and the check for convergence on exact solution is attempted by solving the problem with two different mesh densities. The finite difference energy method is an improvement on the finite difference method – uses the principle of minimum potential energy.

❖ **Numerical Integration**

In this method the second order differential equations be converted to first order differential equations by using the finite difference equations. Initial values are assumed for unknown boundary conditions, and Runge-Kutta method integrates equations across the shell.

❖ **Finite Element**

It uses the element stiffness to get an overall stiffness matrix. The shell is approximated by a number of finite elements – having known displacement polynomials or *Shape Functions.* The stiffness of each element is found out by energy methods – either by closed form solutions or by numerical integration.

$$(F) = (K).\ (\delta) \tag{1.1}$$

(F) Column matrix of forces at node points, (K) is stiffness matrix and (δ) column matrix of nodal displacements.

1.7.3 Flow Induced Vibration Analysis

The first detailed study of flow induced vibrations in bellows was conducted by Gerlach [13]. He observed that the source of fluid excitation was due to vortex shedding from the convolution tips and presented a simplified method of calculating the natural frequencies of bellows under flow induced vibrations. He argued that a bellows with ‘N’ convolutions could be represented as a system of (2N-1) identical masses connected by ‘2N’ identical springs. Relatively, simple expressions were given for elemental masses and springs in terms of bellows geometry. He assumed that the elemental fluid mass could be considered as one-half the mass of the fluid contained between bellows convolutions for the lowest modes of vibration. Gerlach derived an expression for calculating fluid added mass for higher modes of vibration in which the fluid is pressed out and sucked in between convolutions. Then the natural frequencies of axial vibrations were determined by calculating the elemental mass including the added mass and elemental stiffness for a given bellow.

In a subsequent paper, Gerlach [14] developed a concept of *‘Stress Indicator’*- based on the assumptions of linear forced vibration theory to provide an index of the severity of vibrations. Gerlach also suggested that there was no ‘vortex shedding’ occurring in the absence of bellows vibration and compute Strouhal number by equation (2.2) –

$$S = \frac{\omega.L}{V_p}$$

(1.2)

Where 'L' is the length of the free shear layer and is assumed equal to convolution pitch, ω is the response frequency and V_p is the mean flow velocity. Gerlach argued that the characteristic length used for computation shall be convolution pitch instead of convolution width. Using l = 3mm and velocities corresponding to amplitude peaks in four modes, he computed the Strouhal number as 0.45. These results obtained by Gerlach assume ideal up-stream and uniform flow through the bellows. However, for higher flow velocities and non-uniform flow conditions which are observed in actual service, it has been noticed that the resonant peaks occurred at Strouhal number of 0.58 - about 29% higher than that found for uniform flow. Also it is seen from the above equation (2), that as the mean flow velocity (V_p) of the fluid through the bellows increases the value of Strouhal number (S) decreases – and hence increasing the frequency (f). Gerlach also concluded that Strouhal numbers based on the mean flow velocity are not valid.

In 1972, Bass and Holster [6] extended the work of Gerlach to study the vortex excitation of metallic bellows with internal cryogenic flows. It was found that internal cavitation or boiling due to heat transfer in the formation of frost or condensation on outside of bellows convolutions had the effect of damping the vibrations.

Later, Rockwell and Naudascher [41] in their review paper suggested that the actual excitation mechanism in the bellows is probably the free shear layer instability over the periodic cavities created by the bellows convolutions. They also noted that the Strouhal number reported for bellows was less than one-half for a rectangular cavity and speculated that the effect of rounded corners in case of the bellows was to reduce the predominant oscillation frequency.

However, Franke and Carr [12], has shown that damping the upstream and downstream corners of rectangular cavities was very effective in reducing the free shear layer of oscillations. Indirectly, this would mean that coherent shear layer structures should not be made use of over bellows convolutions. This observation was found to be true by Gerlach, if the convolutions were rigid.

The EJMA also provided a simplified method for calculating the natural frequencies of bellows. The approach followed by EJMA is on the assumption that bellows can be considered as a continuous elastic rod. Added mass is accounted for the mass of fluid trapped between the bellows convolutions as was assumed by Gerlach [14] for the lowest modes of vibration. No provision is made in this approach for modes of vibration in which simple translation of fluid between un-deformed convolutions, and hence the assumptions made are not quite reasonable. The EJMA method therefore, may not be very good for estimating the natural frequencies of relatively short bellows or the natural frequencies of higher modes of vibration.

In 1986, Becht [5] has brought out an excellent survey on the developments of numerical and theoretical methods for predicting the response of bellows. The state-of-the–art was assessed covering the earlier work of 1940 and that the directions requiring further development was discussed.

In 1989, Weaver.D.S and Ainsworth. P conducted experiments to demonstrate the fatigue failure of a number of bellows. It was observed that with high internal flow velocities, significant flow induced vibrations developed that led to fatigue failure of bellows.

V.F.Jakubauskas and D.S.Weaver [16] studied the natural vibrations of fluid filled bellows modeling the bellows using axi-symmetric shell finite elements and discretizing the region using axi-symmetric triangular elements. The in-vacuo bellows modes were used as boundary conditions on the potential flow model for the fluid and the added mass determined for each mode of vibration. The added mass thus computed was used to determine the natural frequencies of fluid filled bellows. Results were also obtained by conducting suitable experiments and the theoretical models were verified against experimental results.

In 1997, Jakubauskas and D.S.Weaver [17] studied the transverse vibrations of fluid filled bellows through a theoretical model based on Timoshenko Beam theory and included the effect of added mass on an internal fluid. They developed an analytical expression for comprising the natural frequencies of vibration of bellows in

the form of Raleigh Quotient in a way suitable for hand calculations. The results for the first four modes of vibration are compared with experimental results as well as the results from expressions given by EJMA. It was found that the approach by EJMA was in substantial error, as it neglected the effect of rotatory inertia and convolution distortion component of fluid added mass.

Considering only axial vibrations, Jakubauskas and Weaver (1996) [32] modeled the bellow as a shell of revolution containing ideal fluid and then conducted finite element analysis. The results showed that the EJMA predictions were reasonably good for the lowest modes and more truly for long bellows. However, significant errors were observed, as the bellows became shorter. The effect of convolution shape distortion on fluid added mass was neglected in the EJMA model and so the prediction of natural frequency by this method gave erroneous results.

Having demonstrated the validity of the theoretical model, the investigators examined the degree to which the theoretical model represents an improvement over the existing EJMA model. To this effect, calculations were made for all the above cases using the Bernoulli-Euler theory with and without the contribution of added mass and also using the simplified approach as given in EJMA code.

It was seen that the theoretical calculations using the Bernoulli-Euler theory agreed well with EJMA. However, the transverse frequencies of bellows obtained using EJMA method was an overestimate, because the rotary inertia is neglected and

it was observed that this trend increased with mode number. On the other hand transverse frequencies of bellows in water were seen to be better matching with theoretical results than predicted in air - especially for the first mode.

Hence, Jakubauskas and Weaver concluded the following-

- Excellent agreement between the theoretical methods and experiment.
- Effect of shear component of the bellows neglected in the analysis.
- EJMA model overestimates the transverse natural frequencies of the bellows in both air and water.
- Effect of internal pressurization on bellows natural frequencies is minimal on 1st transverse mode and decreases with increasing mode numbers.
- Fluid flowing through the bellows up to mean fluid velocity of 10m/s has negligible effect on bellows natural frequency.
- Effect of rotary inertia and convolution distortion component of fluid added mass depends on geometry of the bellows.

Jakubauskas [32] studied the axial vibrations of bellows and presented an improved method of computing the added fluid mass for bellows. The added mass, thus computed is shown to consist of three parts – one due to convolution translation in axial direction, the second associated with convolution distortion and the third associated with return flow in central area of cross-section of a bellows. The distortion component for a half convolution was determined using finite element analysis and results were presented for a typical range of bellows geometry.

Azar.R.C, Chadrashekar.S, and Szaban.P [3] developed a computer program in determining the axial vibrations of metallic bellows to improve the performance of bellows by altering the spread of its natural frequencies thereby avoiding resonance. They argued that the earlier techniques for determining the bellows stiffness use either the beam theory or the theory of plates and shells. The first approach according to them was over simplification while the second method derives equations that are difficult to use. Hence, they derived equations based on the Curved Beam Theory. The computer program developed by them uses the Holtzer metho*d* in determining the natural frequencies and mode shapes for the axial vibration. They have also conducted vibration tests on several bellows by fixing one end of the bellows to an Electro-dynamic shaker and the other end free. Results obtained from the computer program and the single degree of freedom model were compared and validated with the experimental results and were found to be satisfactory.

Jakubauskas, V.F [19] derived expressions for column instability. Internal pressure may cause a bellows to become unstable in two different ways in-plane instability and column instability. As the column instability is most frequent case in bellows, the same has been dealt in detail. Expressions are derived for both single and double expansion bellows in lateral and rocking modes respectively.

Jakubauskas, V.F [20] has dealt a new approach of calculating the fluid added mass for regular shaped plates and beams. He strongly advocated that the added mass

term is getting obsolete since any coupled problem of fluid elasticity is now solved directly by considering two mathematical separate problems – of elasticity and hydrodynamics.

Jakubauskas, V.F [29] explained the theory behind the derivation of exact estimation of the natural frequencies for double expansion joints based on classical fixed-fixed end conditions**.** The solution is based on the modified Timoshenko differential equation.

Jakubauskas, V.F [34] briefly explained the work of the first researchers in the area of flow-induced vibrations of the bellows. He also dealt with the various components in the differential equation-boundary conditions and the solution of differential equation. He made a comparison with other methods and found that the expressions for calculation of natural frequencies of bellows based on the Bernoulli-Euler differential equation were not precise for the dynamic calculations of bellows expansion joints. The added mass caused by convolution distortion was found to have a great influence for higher modes. The inside pressure in bellows must also be accounted for. He provided theoretical calculations using the derived formula and confirmed that the predictions of Morashita et.al about the influence of rotary inertia on natural frequencies of bellows were correct.

Jakubauskas, V.F [23] showed that convolution distortion during bending vibrations of bellows, in addition to the very large fluid added mass contained in the

cross section of bellows, can produce even larger added mass effects – especially for shorter bellows OR for higher vibration modes. The solution for the problem is found using the finite element analysis.

Jakubauskas, V.F and Weaver D.S [22] presents the results of an analysis of the fluid added mass in bellows expansion joints during bending vibrations. The added mass consists of two parts – one due to the transverse rigid body motion and the other due to distortion of the convolutions during bending. The distortion component that is neglected in previous works has been considered here and showed to be important for short bellows and higher vibration modes. The distortion component is determined using finite element analysis and also in the graphical form for a range of bellow geometry. The total added mass is given in a form suitable for hand calculations.

Jakubauskas, V.F [25] presents a method to calculate the spring stiffness of a bellow using finite element analysis. The calculation results were then compared with the results obtained from the analytical methods.

1.7.4 Research Work

This section essentially deals with the scope and summary of the research work carried out in the area of vibrations and stability of elastically restrained expansion bellows in pipelines.

1.7.5 Scope of Work

The scope of the present work is to theoretically investigate the effect of elastically restrained ends on the axial and transverse vibrations of expansion bellows. The work carried out is relevant, as the end conditions of bellows in general at industrial sites are quite complex giving rise to unequal rotations at the ends. Physically, the system may be represented as a short pipe nipple welded at both ends of bellows and then the flanges welded to the other end of pipe nipple.

The previous works contributed by researchers Li Ting-Xin et al and Jakubauskas V.F et al. [38,28] show that they had paid more attention to study the effect of classical fixed-fixed type of boundary conditions only and obtained axial and transverse natural frequencies there on. It is also seen that no provision has been made to these effects in EJMA code, while computing the axial and lateral vibration frequencies for single and Universal bellows and so may not be treated as accurate estimates.

The bellows are modeled as an equivalent beam including rotatory inertia as has been considered by earlier researchers and EJMA. The general method followed in deriving exact frequency expressions in the present work is based on “Separation of Variables” approach in order to find the axial and transverse natural frequencies of single bellows. This method is nearly well adopted in analyzing the vibration of Universal (double) bellows also.

The influence and variation of the rotational elastic restraint parameter T, from 10^{-2} to 10^{10} is studied. It is seen that the effect of this parameter on natural frequencies is quite significant. The effect of variation of the internal pressure and flow velocity of fluid in the bellows is also studied. The results obtained thereon by solving the closed form frequency equations are then confirmed numerically by using the finite element method and are found to be in below 10% of error for all the cases.

The particular case of bellows with fixed-fixed boundary condition is a reduced case wherein the elastic restraints against rotation are compared well with the experimental, numerical and theoretical results for both axial and transverse vibrations respectively. All the results are thereon plotted for variation of rotational stiffness, T, internal pressure, P and flow velocity versus axial and transverse frequencies. At various stages of this thesis the results obtained in the chapters have been reported in references [42 to 48] respectively.

CHAPTER 2

ASSUMPTIONS AND DIFFERENTIAL EQUATION OF TRANSVERSE VIBRATION OF BELLOWS

2.1 The Modes of Natural Transverse Vibrations of Bellows

As mentioned earlier in Chapter 1, the U-shaped convoluted type of expansion bellows can vibrate according to longitudinal, shell and beam vibration modes. However, the thesis is devoted to the beam type of vibration mode only.

It is seen that the previous researcher Jakubauskas.V.F, considered the bellows to be welded to pipe ends or flanges that are stiff in comparison to bellows itself and so had assumed that the expansion bellows are fixed at both ends. However, this assumption of considering the bellows fixed-fixed at both the ends result in overestimation of natural frequencies.

Therefore, the aim of the present work is to compute both axial and transverse natural frequencies of single and Universal type of bellows by considering the ends as elastically restrained against rotation.

Even in the case of a Universal bellows expansion joint, that has in between two bellow elements with a smooth connecting pipe, that is generally many times stiffer than the bellows. It is assumed that the pipe is elastic and is not perfectly rigid. Therefore, the system under consideration is represented as a fixed-fixed system with

elastic supports at the ends and a rigid pipe section in the middle. There are two types of transverse vibration modes that can be expected "Lateral" and "Rocking" and are shown in Figures 2.1 & 2.2 respectively.

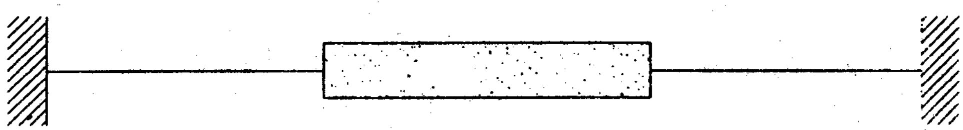


Fig 2.1 Universal Expansion Joint as Elastic System

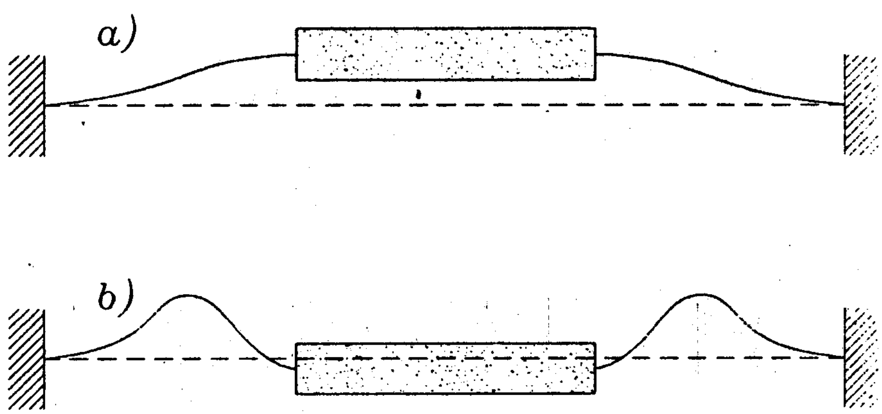


Fig 2.2: Lateral Modes –a) 1st b) 2nd (above)

Rocking Mode –a) 1st b) 2nd (below)

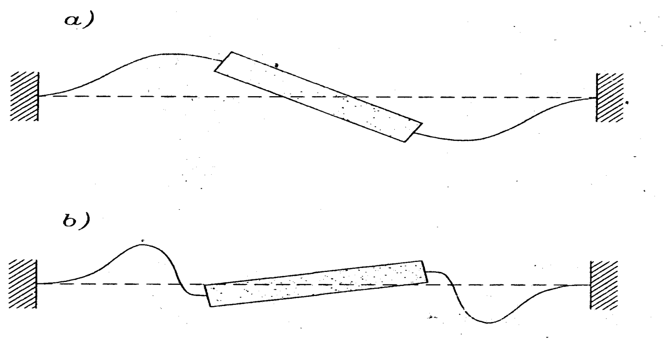


Fig 2.2: Lateral Modes –a) 1st b) 2nd (above)

Rocking Mode –a) 1st b) 2nd (below)

In general, the Double or Universal type of expansion joint vibration problem can be modeled by considering two bending differential equations with eight boundary conditions. But, if it is split in to two separate problems according to the two types of bending modes, the solution can be simplified substantially by considering just one-half, either left or right of the whole system. These two problems are described in subsequent chapters in detail.

2.2 The Differential Equation and General Solution in case of Transverse Vibrations of bellows

The differential equation of vibration of bellows installed in a pipeline –either single or double is given by equation (2.1). For the circular cross-section of a thin beam, k' is the cross-section correction coefficient and is approximately given by Paidoussis et al. (1986) given in equation (2.2)

$$EI\frac{\partial^4 w}{\partial x^4}+m_{tot}\frac{\partial^2 w}{\partial t^2}-\left\{\rho I+\frac{\rho EI}{Gk'}\right\}\frac{\partial^4 w}{\partial x^2 . \partial t^2}+\frac{\rho^2 I}{Gk'}\frac{\partial^4 w}{\partial t^4}=0 \tag{2.1}$$

$$k'=\frac{6(1+\upsilon)\left(1+\gamma^2\right)^2}{(7+6\upsilon)\left(1+\gamma^2\right)^2+(20+12\upsilon)\gamma^2} \tag{2.2}$$

ν = Poisson's ratio

γ = Ratio of internal to external radius of tube

A bellows as a pipe is assumed with following dimensions is considered for analysis –Mean radius R_m = 0.03465m, wall thickness t= 0.00028m and ν=0.25, then R_{INT}

$$\therefore \gamma=\frac{R_{INT}}{R_{EXT}}=0.992$$

=0.03446m and R_{EXT} = 0.03474m

From the above data and according to formula given in equation (2.2), k' = 0.53. We know-

$$G=\frac{E}{2(1+\nu)} \tag{2.3}$$

The shear term coefficient in equation (2.1) becomes –

$$\frac{\rho EI}{Gk'}=2(1+\upsilon)\frac{I}{k'}$$

(2.4)

$$= 2(1+0.25)\frac{I\rho}{0.53} = 4.72\rho I$$

Since the rotatory inertia coefficient in equation (2.1) is equal to ρI, the ratio of shear to rotatory inertia is given by equation (2.5)

$$\left(\frac{\rho EI}{Gk'}\right).\frac{1}{\rho I} = 4.72$$

(2.5)

This shows that for a beam (pipe/tube) the influence of rotatory inertia is 4.72 times less than the influence of shear. Therefore, in short beams i.e. in case of short bellows the problems of rotatory inertia can be neglected.

Now let us consider a bellows as a pipe with convoluted surface as shown in Fig.1.2. Let the bellows have the same dimensions as before with $r_1 = r_2 = 0.00125$m, convolution depth, h = 0.00571m and length of bellows, L=0.00321m respectively. No pressure of 0.0MPa is assumed as atmospheric pressure through out the thesis.

The axial stiffness of a bellows is calculated using the expression given below-

$$k = 4\,R_m\,E\,(t / h)^3 \qquad (2.6)$$

Substituting the values and considering the material of construction of bellow as SS304 Grade stainless steel, E= 2.07 x 10^{11} N/m^2 at ambient conditions –

k = 4 x 0.03465 x 2.07x 10^{11} $(0.00028/0.00571)^3$

Bending stiffness, EI $= ¼\ k\ q\ R_m^2$

$= ¼ \times 3.383 \times 106 \times 0.005 \times 0.03465^2$

$= 5.078\ Nm^2$

$G = 0.828 \times 10^{11}\ N/m^2$ is calculated using the formula given in equation (2.3)

$\therefore \rho EI / G\ k = \rho \times 5.078 / 0.828 \times 10^{11} \times 0.53 = \rho \times 11.57 \times 10^{-11}$ kg-m

The equivalent second moment of area or moment of inertia of bellow cross-section I_b is given by –

$$I_b = \frac{\pi R_m^3 t.(2\pi r_1 + 2L)}{4 r_1} \tag{2.7}$$

Where, 't' is the thickness of bellows

$$= \frac{\pi \times 0.034653 \times 0.00028\ (2\pi \times 0.00125 + 2x\ 0.00321)}{4 \times 0.00125}$$

$$= 1.045 \times 10^{-7}\ m^4$$

$\therefore$ The ratio of shear rotatory inertia is as follows-

$$\left(\frac{\rho EI}{Gk'}\right).\frac{1}{\rho I_b}$$

$$= \frac{\rho \times 11.57 \times 10^{-11}}{\rho \times 1.045 \times 10^{-7}} = 11.072 \times 10^{-4}$$

This ratio shows that for a corrugated pipe (bellows) with a moderate convolution depth, the influence of shear coefficient is just a small fraction, 0.00111 of the rotatory inertia –without taking into account the fluid rotatory inertia. Therefore, for the investigation of transverse vibrations of bellows, the influence of shear can be ignored, and only the rotatory inertia is accounted for.

Let us now consider the coefficient of the last term in the differential equation (2.1),

$$\frac{\rho^2 I}{Gk'} = \frac{5.078\rho^2}{EG.k'} \tag{2.8}$$

It is seen that as the value of EG in the denominator is large compared to the numerator and so this coefficient is very small in comparison with coefficients of other terms in equation (2.1) and so is ignored.

The influence of rotatory inertia on the frequency of natural vibrations of bellows is studied. For the sake of comparison of natural frequencies with the results obtained by Jakubauskas.V.F, bellows having the same geometrical dimensions are considered here too.

The differential equation (2.1) without the shear term is –

$$EI\frac{\partial^4 w}{\partial x^4} + m_{tot}\frac{\partial^2 w}{\partial t^2} - \left\{\rho I + \frac{\rho EI}{Gk'}\right\}\frac{\partial^4 w}{\partial x^2 . \partial t^2} = 0$$

$$J = \left\{ \rho I + \frac{\rho EI}{Gk'} \right\}$$

(2.9)

Where J is the mass moment of inertia of bellows cross-section or rotatory inertia per unit length and m_{tot} is the total mass of the bellows that includes the bellow material mass and mass of the fluid flowing through the bellows.

Let us assume the first mode for simply supported ends to be defined by equation (2.10)-

$$w = A \sin \pi \frac{x}{L} \cos \omega_1 t$$

(2.10)

Substituting equation (2.9) into the differential equation (2.8) we get –

$$EI \left(\frac{\pi}{L} \right)^4 - \omega_1^{\,2} J \left(\frac{\pi}{L} \right)^2 - \omega_1^{\,2} m_{tot} = 0$$

(2.11)

Leads to the frequency expression that is as follows –

$$\omega_1 = \left(\frac{\pi}{L} \right)^2 - \sqrt{\left[\frac{EI}{m_{tot}} \right]} \sqrt{\left(\frac{1}{1 + \frac{J}{m_{tot}} \left(\frac{\pi}{L} \right)^2} \right)}$$

(2.12)

$$K = \sqrt{\left(\frac{1}{1 + \frac{J}{mtot} \left(\frac{\pi}{L} \right)^2} \right)}$$

Where K is the coefficient that takes care of the rotatory inertia of bellows. If J=0. K=1 and the frequency equation becomes well known frequency equation of Bernoulli-Euler equation. In order to compute the value of K, it is required to calculate J and total mass

m_{tot}.

$$J = \frac{\pi R_m^3\, t\,(\pi r_1 + \pi r_2 + 2l)}{2(r_1+r_2)}\rho_b + \frac{\pi R_m^3\,(l+r_1+r_2)\,(2r_2-t)}{2(r_1+r_2)}\rho_f \qquad (2.13)$$

If $r_1=r_2=0.00125$, bellows length L = 0.0693m, ρ_b= 7560kg/m^3 and ρ_f = 1000kg/m^3 and substituting all the values in the above equation (2.13) we get J = 0.001153kg-m.

The total mass of the bellows is calculated according to equation (2.14) –

$$m_{tot} = \frac{\pi R_M\,(\pi\, r_1 + \pi\, r_2 + 2L)}{r_1 + r_2}\rho_b + \pi R_m^2\,\rho_f \qquad (2.14)$$

Substituting the values above we get m_{tot} = 5.139 kg/m.

Now substituting the numerical values for J and m_{tot} in the expression – we get the value of as K = 0.827. Since the value of K is less than unity, it is inferred that the inertia of rotation of the cross-section lowers the natural frequency in comparison with the Bernoulli-Euler solution by about 17.3% for simply supported bellows. A similar work is carried out for fixed-fixed elastically restrained ends against rotation.

2.3 The Influence of Internal Pressure and the Centrifugal force of flow on Transverse Vibrations of Bellows

The differential equation is derived that takes into account the static inside pressure and the centrifugal force of flow of fluid in addition to the Bernoulli-Euler conditions. The differential element of such a pipe is shown in Fig 2.1 The moment equation for bellows with respect to point O is given as –

$$\frac{\partial M}{\partial x}.dx - Q.dx + P\pi Rm^2 dw = 0$$

(2.15)

From where the shear force equation Q of bellows is obtained as follows –

$$Q = \frac{\partial M}{\partial x} + P\pi Rm^2 \frac{\partial w}{\partial x}$$

(2.16)

The translational differential equation with respect to the y direction is-

$$m_b \frac{\partial^2 w}{\partial t^2}.dx = -\rho_f.A_{min}.V^2 \frac{\partial^2 w}{\partial x^2}.dx - \frac{\partial Q}{\partial x}dx$$

(2.17)

$$m_b \frac{\partial^2 w}{\partial t^2}.dx + \rho_f.A_{min}.V^2 \frac{\partial^2 w}{\partial x^2}.dx + \frac{\partial Q}{\partial x}dx = 0$$

(2.18)

A_{min} is the cross-section of the bellows and is given by-

$$A_{min} = \pi \{R_m - L/2 - r_1 - t/2\}^2 \qquad (2.19)$$

From beam theory, we know –

$$M = EI \frac{\partial^2 w}{\partial x^2} \qquad (2.20)$$

Substitution of equation (2.20) into equation (2.15) and subsequently substituting the equation (2.15) into equation (2.17) yields the differential equation for transverse vibration-

$$EI \frac{\partial^4 w}{\partial x^4} + P\pi Rm^2 + \rho_f A_{min} V^2 \frac{\partial^2 w}{\partial x^2} + m_b \frac{\partial^2 w}{\partial t^2} \qquad (2.21)$$

Therefore, it is seen from the above equation that both the pressurization and centrifugal force effects on the vibrating bellows are similar to that of both coefficients of the curvature term $\partial^2 w / \partial x^2$

$$EI \frac{\partial^4 w}{\partial x^4} + \eta \frac{\partial^2 w}{\partial x^2} + m_b \frac{\partial^2 w}{\partial t^2} \qquad (2.22)$$

Where $\eta = P\pi R^2_m$ the pressure coefficient or $\rho_f A_{min} V^2$ the centrifugal coefficient or both taken together.

Division of the above equation (2.22) by EI gives-

$$\frac{\partial^4 w}{\partial x^4} + \frac{\eta}{EI}\frac{\partial^2 w}{\partial x^2} + \frac{m_b}{EI}\frac{\partial^2 w}{\partial t^2} = 0 \qquad (2.23)$$

If

$$a' = \sqrt[4]{\frac{m_b}{EI}} \qquad c = \sqrt{\frac{\eta}{2EI}}$$

Then the differential equation 2.23 can be written as follows -

$$\frac{\partial^4 w}{\partial x^4} + 2c^2 \frac{\partial^2 w}{\partial x^2} + a^4 \frac{\partial^2 w}{\partial t^2} = 0 \qquad (2.24)$$

Let w = X (x). T (t)

Then, if T (t) is some harmonic function, the derivatives of (2.24) needed in equation (2.23) are as follows-

$$\frac{\partial^4 w}{\partial x^4} = T(t)\frac{dX^4}{dx^4}$$

$$\frac{\partial^2 w}{\partial x^2} = T(t).\frac{dX^2}{dx^2}$$

And $$\frac{\partial^2 w}{\partial t^2} = -\omega^2 X\,.T(t)$$

Substituting the above derivatives into equation (2.24) gives

$$\frac{d^4 X}{d x^4} + 2c^2 \frac{d^2X}{dx^2} + \lambda^4 .\, X = 0 \qquad (2.25)$$

Where $\lambda^4 = a'^4\ \omega^2$ (2.26)

Let $X = C\ e^{sx}$

$$\frac{d^4X}{d\,x^4} = Cs^4\,e^{sx} \quad \text{and} \quad \frac{d^2X}{dx^2} = Cs^2\,e^{sx} \tag{2.27}$$

Substitution of the above derivatives into equation (2.24) we get a quartic equation

$s^4 + 2c^2s^2 - \lambda^4 = 0$ and the roots of which are as follows –

$s_{1,2} = \pm\,\alpha$
$s_{3,4} = \pm\ i\beta$

Where $\alpha = \sqrt{-c^2 + \sqrt{c^4 + \lambda^4}}$ (2.28)

$\beta = \sqrt{c^2 + \sqrt{c^4 + \lambda^4}}$ (2.29)

Solution of the equation (2.24) is written as –

$$X = C_1\,e^{a'x} + C_2\,e^{-a'x} + C_3\,e^{i\beta x} + C_4\,e^{-i\beta x} \tag{2.30}$$

$$C_1 = \frac{B + A}{2}$$

$$C_2 = \frac{B - A}{2}$$

$$C_3 = \frac{D - iC}{2}$$

$$C_4 = \frac{D + iC}{2}$$

$\therefore$ Equation (2.30) can be written as follows-

$$X = A \sinh \alpha x + B\cosh \alpha x + C\sin\beta x + D\cos \beta x \tag{2.31}$$

And the derivative of which becomes

$$\frac{dX}{dx} = A\alpha \cosh \alpha x + B\alpha \sinh \alpha x + C\beta \cos\beta x - D\beta \sin \beta x \tag{2.32}$$

The procedure adopted to derive the solution of differential equation is given by considering an example for fixed-fixed case of bellows and is assumed to be same as pipe. The four required boundary conditions are given by-

At X=0

$$X(0) = 0$$

$$\frac{dX(0)}{dx} = 0$$

At X=L

$$X(L) = 0$$

$$\frac{dX(L)}{dx} = 0$$

Substitution of the boundary conditions results in a system of linear simultaneous equations with respect to A, B, C &D respectively.

$$B + D = 0 \tag{2.33}$$

$$A\alpha + C\beta = 0 \tag{2.34}$$

$$A \sinh \alpha L + B \cosh \alpha L + C \sin\beta L + D \cos\beta L = 0 \tag{2.35}$$

$$A\alpha \cosh\alpha L + B\alpha \sinh \alpha L + C\beta \cos\beta L - D\beta \sin\beta L = 0 \tag{2.36}$$

For a non-trivial solution, the determinant formed by the coefficients of this equation must be equal to zero-

$$\begin{vmatrix} 0 & 1 & 0 & 1 \\ \alpha & 0 & \beta & 0 \\ \sinh \alpha L & \cosh \alpha L & \sin \beta L & \cos \beta L \\ \alpha \cosh \alpha L & \alpha \sinh \alpha L & \beta \cos \beta L & -\beta \sin \beta L \end{vmatrix} = 0 \tag{2.37}$$

The expansion of the determinant will result in the frequency equation of the differential equation –

$$(\beta/\alpha - \alpha/\beta) \sinh \alpha L \,.\, \sin \beta L + 2 \cosh \alpha L \,.\, \cos\beta L - 2 = 0 \tag{2.38}$$

The influence of inside pressure is also investigated –

$$\eta = P\pi R_m^2 \tag{2.39}$$

According to EJMA, the critical pressure of bellows is

$$Pcr = \frac{\pi k q}{L^2}$$

Where k is the axial stiffness of the bellows and q is the pitch

$$\therefore P_{max} = P_{cr} / 6.666 = \pi k q / 6.666 L^2 \tag{2.40}$$

We know that $EI = ¼ k q R_m^2$ (2.41)

Substituting equation (2.40) into equation (2.41) and subsequently into expression for ‘c’-

$c = 0.5477\,\pi / L$

Bellows length, $L = 2\,R_m$. So in this case $L = 0.0693$m

$\therefore c = 24.831\ L/m^2$

Now, using the numerical value of c and expressions α and β the frequency equation (2.38) can be solved numerically.

The first frequency gave $\lambda = 65.61205$

From 2.26, $\omega = \lambda^2 / a'^2$

Substituting the expression for, a' and the calculated value of λ into the above equation – the first mode frequency of the bellows for fixed-fixed case is calculated –

$$\omega = 4305 \sqrt{\{EI / m_b\}}$$

The natural frequency of bellows when inside and outside pressure difference is equal to zero, then the second term in the differential equation (2.24) is cancelled and the simplified equation for the first mode of fixed-fixed beam becomes –

$$\omega = 4.73^2 / L^2 \sqrt{\{EI / m_b\}}$$

For Bellows length, $L = 0.0693$m: $\omega = 4658.68 \sqrt{\{EI / m_b\}}$

So comparison of both the frequencies shows that taking into account the maximum pressure allowed by EJMA standard, the natural frequency becomes lower by about 7.6%

CHAPTER 3

AXIAL VIBRATIONS OF SINGLE BELLOWS EXPANSION JOINT RESTRAINED AGAINST ROTATION

3.1 Investigations of Axial Vibrations in Bellows

The previous work contributed by Li Ting-Xin et al, in the area of axial vibrations of bellows dealt with classical fixed-fixed end conditions. The present work considers the ends as elastically restrained against rotation and aims at finding out the effect of the rotational restraint parameter, T on the axial natural frequencies. The analysis considers finite stiffness axial restraints on the bellows, i.e., solving the set of equations with non-homogeneous boundary conditions. The bellow specimen considered here for comparison has the same dimensions as taken by Li Ting-Xin in his analysis. The transcendental frequency equation is derived and exact as the first, second and third mode frequencies computed are in close agreement to the ones obtained by Li Ting-Xin.

The paper contributed by Li Ting-Xin,et al [38], presents equations to calculate the axial and lateral natural frequencies of single bellows with three types of end conditions- one end fixed and other end free, one end fixed and other end attached to weight and both ends of bellows fixed. The theoretical results were then compared with experiments performed on bellow specimens having different geometrical parameters. Though the error corresponding to the experimental value was found to be reasonably close to theoretical value, the end conditions represented

by Li Ting-Xin do not represent most of the practical situations. It is seen on several occasions that bellows ends are welded to a small pipe spool that has a lumped mass such as a valve or an instrument. The physical system considered for the present work in axial vibrations is shown in Fig 3.1. A weight W_o is attached as shown and is connected to a spring of stiffness R_1. It is assumed that a straight pipe is partially fixed at the bottom and connected to a spring of stiffness R_2. The various geometrical parameters of the U shaped bellows used in the analysis are shown in Fig 1.2.

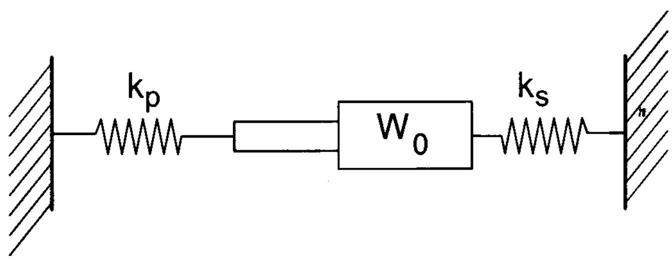


Fig 3.1: Physical System showing End Supports Elastically Restrained

The geometrical dimensions of U shaped bellows are given below in Table 3.1. In order to compare the frequencies, the same geometrical dimensions taken by Li Ting-Xin are considered here [38].

Table 3.1 Geometrical Dimensions of bellows (mm)

D_b	h	q	n	t	N
322.5	24.5	22.4	1	0.49	9

Where D_b is outside diameter of bellows, h –depth of convolution, q –pitch, n-number of plies, t-thickness of bellows, n-number of plies and N-number of convolutions respectively.

3.2 Differential Equation for Axial Vibration of Bellows

The bellows are considered as straight pipe. The differential equation to express the axial vibration for the straight pipe is given by -

$$\frac{\partial^2 u}{\partial t^2} = a^2 \cdot \frac{\partial^2 u}{\partial x^2} \tag{3.1}$$

Where 'u' is the axial displacement of the pipe (mm) and

$$a = \sqrt{\frac{E.g}{\upsilon}} \tag{3.2}$$

't' is the time (s), E is the elastic modulus of bellows material (MPa), g is the gravitational acceleration (9806.65mm s^{-2}), ν is the weight per unit volume of the bellows material (N/mm^3) respectively.

3.3 Derivation of Natural Frequency Equation for Axial Vibrations of Elastically Restrained Single Bellows

Considering the axial vibrations of a single bellows expansion joint the two boundary conditions for the system are as follows-

We know,

$$\text{At } x = 0, \left.\begin{matrix} u(x) = 0 \\ u(t) = 0 \end{matrix}\right\} \tag{3.3}$$

$$\text{At } x = 0 \qquad AE \frac{\partial u}{\partial x} = k_p u \tag{3.4}$$

At x = L $\qquad AE\frac{\partial u}{\partial x}+\frac{W_o}{g}\frac{\partial^2 u}{\partial t^2}+k_s u=0$ (3.5)

The exact solution of the differential equation is written as follows –

$u(x, t) = \{C \sin \beta + D \cos \beta x\}.e^{i\omega t}$ (3.6)

Where $\beta = \omega_i / a$ (3.7)

And ω_i is circular frequency in radian/s, C & D are integration constants. Now applying the boundary conditions and substituting t=0 we get –

$u(x) = C \sin \beta x + D \cos \beta x$ (3.8)

$$\frac{\partial u(x)}{\partial x}=\beta(C\cos\beta x-D\sin\beta x)$$

(3.9)

Applying the first boundary condition and substituting for $\partial u/\partial x$ and u (x) in equation (3.4), we get –

$$\therefore AE\frac{\partial u}{\partial x}=kp.u$$

(3.10)

$$\frac{\partial u(x)}{\partial x}=\beta(C\cos\beta x-D\sin\beta x)$$

(3.11)

$AE\,\beta(C \cos \beta x – D \sin \beta x) = k_p(C \sin\beta x + D \cos\beta x)$ (3.12)

At x =0, i.e.

$$AE\frac{\partial u(x=0)}{\partial x} = kp.u$$

(3.13)

$\therefore$ Substituting x=0 in equation (3.13) we get,

$(\beta.\ AE)\ C = k_p.\ D$ (3.14)

$$\therefore C = \left(\frac{kp}{\beta.AE}\right)D$$

(3.15)

Now applying the second boundary condition and substituting for u (x), $\partial u/\partial x$ and $\partial^2 u/\partial t^2$ in equation (3.5) we get-

$$\because u(x,t) = u(x)\,e^{i\omega t}$$

(3.16)

$$AE.\beta(C\cos\beta x - D\sin\beta x) + \left(k_s - \frac{Wo\,\omega^2}{g}\right)(C\sin\beta x + D\cos\beta x) = 0$$

(3.17)

At x= L equation (3.17) becomes-

Where L is the bellows length we get-

$$AE.\beta(C\cos\beta L - D\sin\beta L) + \left(ks - \frac{Wo\,\omega^2}{g}\right)(C\sin\beta L + D\cos\beta L) = 0$$

$$AE.\beta(C\cos\beta x - D\sin\beta x) - \frac{Wo\,\omega^2}{g}(C\sin\beta x + D\cos\beta x) +$$
$$ks(C\sin\beta x + D\cos\beta x) = 0$$

(3.18)

$$C\left[AE.\beta\cos\beta L + \left(ks - \frac{Wo\,\omega^2}{g}\right)\sin\beta L\right] = D\left\{AE\beta\sin\beta L - \left(ks - \frac{Wo\,\omega^2}{g}\right)\cos\beta L\right\}$$

(3.19)

$$C\left\{\left[AE.\beta\cos\beta L + \left(ks - \frac{Wo\,\omega^2}{g}\sin\beta L\right)\sin\beta L\right]\right\} = D\left\{AE\beta\sin\beta L - \left(ks - \frac{Wo\,\omega^2}{g}\right)\cos\beta L\right\}$$

(3.20)

Substituting (3.15) in (3.20) we get -

$$\left(\frac{kp}{\beta.AE}\right)D\left\{\left[AE.\beta\cos\beta L + \left(ks - \frac{Wo\,\omega^2}{g}\sin\beta L\right)\right]\right\} = D\left\{AE\beta\sin\beta L - \left(ks - \frac{Wo\,\omega^2}{g}\right)\cos\beta L\right\}$$

(3.21)

Cancellation of D on both sides, we get-

$$\left(\frac{kp}{\beta.AE}\right)\left\{\left[AE.\beta\cos\beta L + \left(ks - \frac{Wo\,\omega^2}{g}\sin\beta L\right)\right]\right\} = \left\{AE\beta\sin\beta L - \left(ks - \frac{Wo\,\omega^2}{g}\right)\cos\beta L\right\}$$

(3.22)

Now multiplying throughout by AEβ we get the final frequency equation as-

$$\tan\beta L = \frac{AE\beta\left(kp + ks - \dfrac{Wo\,\omega^2}{g}\right)}{kp\left(ks - \dfrac{Wo\,\omega^2}{g}\right) + (\beta AE)^2}$$

(3.23)

$$\tan\beta L = \frac{\beta\left(\dfrac{kp}{AE} + \dfrac{ks}{AE} - \dfrac{Wo\,\omega^2}{g.AE}\right)}{\beta^2 + \dfrac{kp}{AE}\left(\dfrac{ks}{AE} - \dfrac{Wo\,\omega^2}{g.AE}\right)}$$

(3.24)

$$\tan\beta L = \frac{AE\beta\left(kp + ks - \dfrac{Wo\,\omega^2}{g}\right)}{(\beta AE)^2 + kp\left(ks - \dfrac{Wo\,\omega^2}{g}\right)}$$

(3.25)

Dividing throughout by AE we get-

$$\tan\beta L = \left\{\frac{\beta L\left(\dfrac{kp.L}{AE} + \dfrac{ks.L}{AE} - \dfrac{Wo.L\,\omega^2}{g.AE}\right)}{(\beta L)^2 + \dfrac{kp.L}{AE}\left(\dfrac{ks.L}{AE} - \dfrac{Wo.L\,\omega^2}{g.AE}\right)}\right\}$$

(3.26)

Let

$$T_p = \frac{k_p.L}{AE} \qquad T_b = \frac{k_s.L}{AE}$$

(3.27)

Where T_p and T_b are the rotational restrained parameters of the pipe and bellows respectively and are non-dimensionless. Also, we refer T_p, T_b as T_1 and T_2 and k_p, k_s as R_1 and R_2 in the subsequent chapters of the thesis, all meaning the same.

$$(\beta L) = \omega i\left(\frac{L}{a}\right) = \sqrt{\frac{LA\upsilon}{AE\,\frac{g}{L}}} = \sqrt{\frac{\frac{G}{g}}{\frac{Q}{\partial u/\partial x}.L}}$$

And we know,

$$\beta = \frac{\omega_i}{a} \quad \& \quad a = \sqrt{\frac{Eg}{\upsilon}}$$

(3.28)

Where

G=LAν and

$$k = \frac{Q}{\partial u/\partial x}.L \qquad \alpha = \frac{G}{Wo}$$

Where α is the ratio of weight of bellows to the weight attached in pipeline, G is the total weight of bellow material with fluid in KG and k is the axial spring rate of pipe in (N/m).

$$\therefore (\beta L) = \omega_i\sqrt{\frac{G}{g.k_P}}$$

(3.29)

3.4 Results for Different End Conditions

Case 1 – The frequency equation for one end of bellows fixed and elastically restrained and the other end free (R_2=0, T_b=0, W_o=0 & $R_1 = \infty$) is given by-

$$\tan \beta_i = \frac{\beta_i \{(\alpha(T_b + T_p) - \beta_i^2)\}}{\{\alpha (1 - T_b T_p) + T_p \beta_i^2\}}$$

$$\beta_i = \omega_i.L\sqrt{\rho / E}$$

$$T_p = \frac{R_1.L}{E.A}$$

$$T_b = \frac{R_2.L}{E.A}$$

$$\beta_i \tan \beta_i = \alpha \qquad (3.30)$$

We know,

$$f_i = \frac{\omega_i}{2\pi} \qquad (3.31)$$

$$\therefore f_i = 49.5 (i - 0.5) \sqrt{k_n} / G \qquad (3.32)$$

Where T_b and T_p are dimensionless rotational restraint parameters, W_o is the weight attached to the pipe and R_1 and R_2 are rotational stiffness of pipe and spring

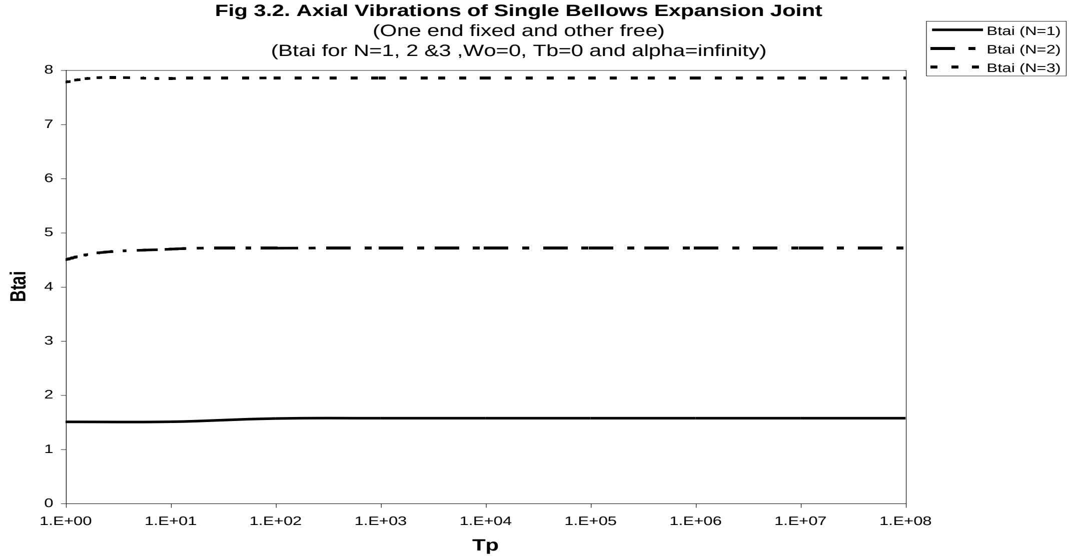
respectively. The values of β_i for the modes of vibration N=1, 2 & 3 and increasing order of T_p from 0.01 to 10^9 are found out and the frequency equation solved.

A computer program in FORTRAN is developed to find the solution of the frequency equations and compute the value of β_i. The values of β_i for the modes of vibration N=1, 2 & 3 and increasing order of T_p is given in Table 3.2.

Table 3.2: Case 1 (One end fixed and other Free (k_s=0, T_s=0, W_o=0 & $T_p = \propto$)

T_p	β_i, N=1	β_i, N=2	β_i, N=3
0.01	1.502	3.17331	6.3465
0.1	1.5023	3.4761	6.8862
1.0	1.5022	4.4934	7.7725
10.0	1.5044	4.6910	7.8412
10^2	1.5644	4.71026	7.8527
10^3	1.5701	4.71217	7.853
10^4	1.5707	4.71236	7.8539
10^5	1.5707	4.71238	7.8539
10^6	1.5707	4.71238	7.8539
10^7	1.5707	4.71238	7.8539
10^8	1.5707	4.71238	7.85398
10^9	1.5707	4.7123	7.8539

Fig 3.2. Axial Vibrations of Single Bellows Expansion Joint
(One end fixed and other free)
(Btai for N=1, 2 &3 ,Wo=0, Tb=0 and alpha=infinity)



The value of β_i obtained at $T_p = \propto$ is 1.5707 for N=1; 4.7123 for N=2 and 7.8539 for N=3 respectively. For Case 1, W_0=0, α=0 and R_2 and T_s=0. It is seen that

as the value of elastic restraint T_p of the pipe increases from 0.01 to 10^9, the frequencies tend to increase for all the modes of vibration. As the restraint parameter T approaches ∞, the frequency increases by about 54% for N=1 (P=0.0MPa) and by 69% for N=1 (P=166.0MPa) respectively. However, it is observed that there is no change in frequency and it becomes constant from $T=10^4$ onwards. Table 3.3 gives the values of axial frequencies in air and water. P is the internal pressure in the bellow and 0.0MPa represents atmospheric pressure and P=166.0MPa is the maximum critical pressure up to which the bellows can be subjected.

Table 3.3 Axial frequencies in Air and Water for Case 1

SP	Air			Water		
	Hz			Hz		
	N=1	N=2	N=3	N=1	N=2	N=3
1	40.5	121.5	202.5	25.5	79.95	124.5

Case 2 – The frequency equation for both ends of the bellows elastically restrained ($W_o=0$ and $R_2=\infty$) is given by-

$$\tan \beta_i = \frac{\beta_i \{((1/T_b + 1/T_p) - \beta_i^{2}) / \alpha (T_b . T_p\}}{\{[1/(T_b . T_p) - 1] + (\beta_i^2 / T_b . \alpha)\}}$$

$$f_i = 49.5i * \sqrt{k_n/G} \qquad (3.33)$$

Where 'i' is the order number of frequency, i = 1, 2, 3.. The values of β_i for the mode numbers N=1, 2 & 3 and different values of T_p are given in Table 3.5.

The values of β_i for the mode numbers N=1, 2 & 3 and different values of T_p (0.01 to 10^9) are obtained. The value of β_i at $T_p = \propto$ is 3.1415 for N=1, 6.2831 for N=2 and 9.4247 for N=3 respectively. For Case2, W_0=0, $\alpha = \propto$ and R_2 & $T_b = \propto$. The axial natural frequencies for this case are presented in Table 3.4.

Table 3.4 Axial frequencies in Air & Water for Case 2

SP	Air			Water		
	Hz			Hz		
	N=1	N=2	N=3	N=1	N=2	N=3
1	81.0	162.5	243.0	51.1	101.2	149.4

Table 3.5 Case 2 (Both ends fixed W_o=0 & $T_b = \propto$)

T_p	β_i, N=1	β_i, N=2	β_i, N=3
0.01	1.57713	4.71451	7.85525
0.1	1.63199	4.733518	7.86669
1.0	2.02875	4.91318	7.97866
10.0	2.86277	5.76055	8.70831
10^2	3.11049	6.22105	9.33172
10^3	3.13845	6.27690	9.41536
10^4	3.14127	6.28257	9.42383
10^5	3.14156	6.28312	9.42468
10^6	3.14158	6.28317	9.42476
10^7	3.14159	6.28318	9.42477
10^8	3.14159	6.28318	9.42477
10^9	3.14159	6.28318	9.42477

The value of β_i for elastically fixed-fixed condition and for $T_p \Rightarrow \infty$ is 3.1415 for N=1, 6.2831 for N=2 and 9.4247 for N=3 respectively. The weight W_0=0, $\alpha = \propto$ and k_s & $T_s = \propto$.

Case 3 – One end fixed and other end attached to weight represents a situation where one end of the bellows is welded to a pipe and the other end is attached to a lumped weight W_0 ($W_0 \neq 0$, and R_2=0) either in form of a valve or an instrument. Therefore,

the values of β_i for this type of end condition are different from the other two cases and are obtained by varying T_p and keeping $1/\alpha$ (0.1, 1, 100 & 1000) a constant value. It depends on the value of α. The frequency equation for this case is given by-

$$\tan \beta_i = \frac{\beta_i \{T_b + T_p - (1/\alpha)\, \beta_i^{\,2}\}}{\{1 - T_b . T_p + (1/\alpha)\, T_p . \beta_i^2\}}$$

The values of β_i are found for different values of $1/\alpha$ and are as follows.

For $1/\alpha$ =0.1, β_i =3.14159 (N=1), 6.28318 (N=2) & 9.4247 (N=3)

For $1/\alpha$ =1.0, β_i =3.14159 (N=1), 6.28318 (N=2) & 9.4247 (N=3)

For $1/\alpha$ =100, β_i =3.14159 (N=1), 3.14159 (N=2) & 6.28318 (N=3)

For $1/\alpha$ =1000, β_i =1.5671 (N=1), 3.14159 (N=2) & 6.28318 (N=3)

By substituting the values of β_i in the frequency equation, the axial natural frequencies are obtained.

Table 3.6 Axial frequencies in Air & Water for Case 3

SP	α	Air			Water		
		Hz			Hz		
		N=1	N=2	N=3	N=1	N=2	N=3
1	0.1&1.0	81.0	162.5	243.0	51.1	101.2	149.4
1	100&1000	40	81	161	25	50	99.5

Table 3.4 presents axial frequencies for different values of $1/\alpha$ = 0.1, 1.0, 100 and 1000 respectively. It is seen that the values of β_i in both the cases tend to be constant from $T_p = 10^5$ onwards. However, for lower values of $1/\alpha$ of 0.01 and 0.001, which means the bellows weight is greater than the weight attached (G>Wo), the values of β_i almost become constant for N=2 & 3 and vary for N=1 only. Axial

natural frequencies in air and water are obtained for specimen number 1 by exact are given in Table 3.6. It is seen that the frequencies obtained by exact method has a percentage of error of 7.2% and 7.4% in air and water respectively in comparison to the experimental and analytical results obtained by Li Ting's on specimen (SP-1) [15].

Table 3.7 Comparison of Axial Frequencies in Hz

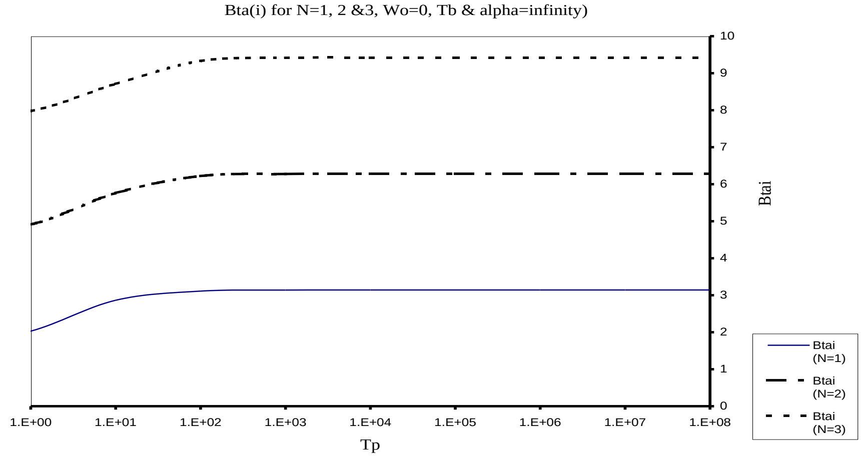
Air

SP	EXP [38]	Li Ting [38]	Exact [42]	%error
1	37.5	40.4	40.5	7.4

Water

SP	EXP [38]	Li Ting [38]	Exact [42]	%error
1	27.5	27.5	25.5	7.2

Fig 3.3. Axial Vibrations of Single Bellows Expansion Joint
(Both ends elastically fixed)
Bta(i) for N=1, 2 &3, Wo=0, Tb & alpha=infinity)



Tables 3.8 & 3.11 present β_i values for $1/\alpha$ of 0.1 and 1.0 respectively. It is seen from Figures 3.5 & 3.6 that the values of β_i in both the cases tend to be constant from $T_p = 10^5$ onwards. However, for lower values of $1/\alpha$ of 0.01 and 0.001, which means the

bellows weight is greater than the weight attached (G>Wo), the values of β_i are almost constant for N=1 & 2 and vary for N=1 only. The same is shown in Figures 3.7 & 3.8 respectively.

Table 3.8 Case 3
(One end fixed and other attached to weight
$1/\alpha$=0.1 and $T_b = \infty$)

T_p	β_i, N=1	β_i, N=2	β_i, N=3
0.01	1.435	4.3076	7.2289
0.1	1.4899	4.3239	7.2366
1.0	1.8964	4.4899	7.3172
10.0	2.8417	5.5990	8.228
10^2	3.1101	6.2185	9.323
10^3	3.13845	6.2768	9.4152
10^4	3.14127	6.2825	9.4238
10^5	3.14156	6.28312	9.4246
10^6	3.14158	6.28317	9.4247
10^7	3.14159	6.28318	9.4247
10^8	3.14159	6.28318	9.4247

Fig 3.4. Axial Vibrations of Single Bellows Expansion Joint
(One end of bellows fixed and other end attached to weight)
(Btai for N=1, 2&3) and alpha=10)

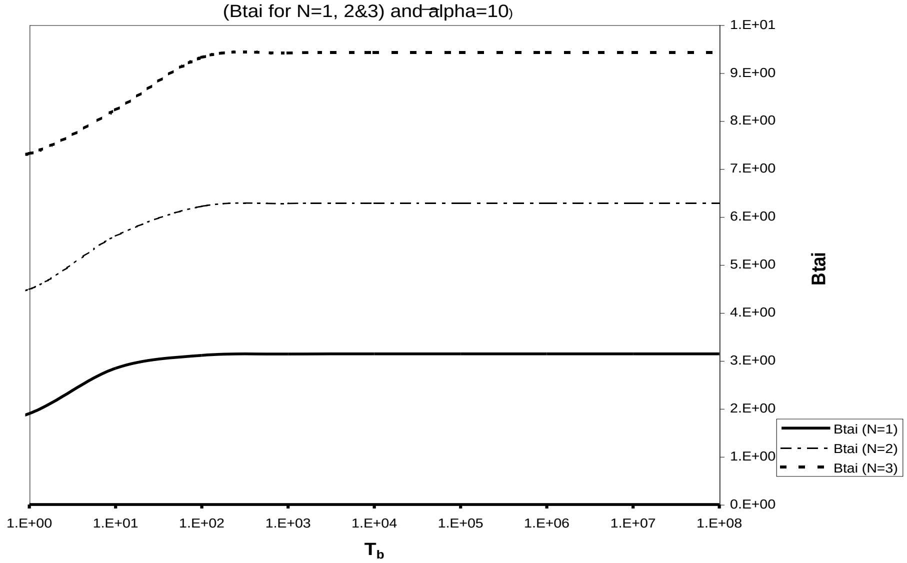


Fig 3.5. Axial Vibrations of Single Bellows Expansion Joint
(One end fixed and other end attached to weight)
(Btai for N=1,2&3 and alpha =1.0)

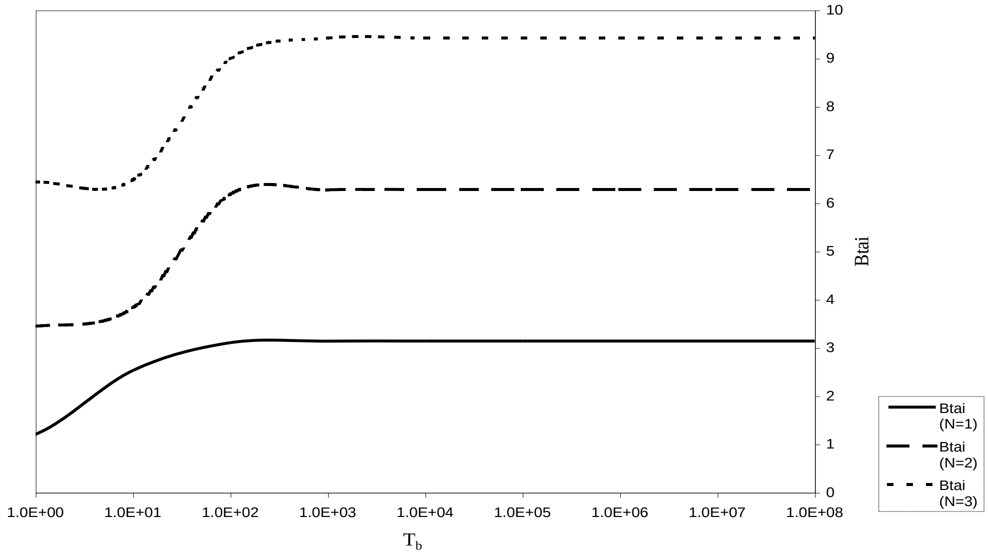


Table 3.9 Case 3
(One end fixed and other attached to weight
$1/\alpha$=1.0 and T_b=$\propto$)

T_p	β_i, N=1	β_i, N=2	β_i, N=3
0.01	0.8645	3.42583	6.43733
0.1	0.9014	3.42775	6.43765
1.0	1.2077	3.44823	6.44095
10.0	2.5293	3.82916	6.48289
10^2	3.1072	6.18340	8.98748
10^3	3.1384	6.27665	9.42382
10^4	3.14127	6.28312	9.42468
10^5	3.14156	6.28317	9.42468
10^6	3.14158	6.28317	9.42476
10^7	3.14159	6.28318	9.42477
10^8	3.14159	6.28318	9.42477

Table 3.10 Case 3
(One end fixed and other attached to weight
$1/\alpha=100$ and $T_b=\propto$)

T_p	β_i, N=1	β_i, N=2	β_i, N=3
0.01	0.1003	3.1447	6.28477
0.1	0.1047	3.1447	6.28477
1.0	0.1411	3.1447	6.28477
10.0	0.3311	3.1448	6.28478
10^2	1.0031	3.1455	6.28481
10^3	3.0809	3.2236	6.28531
10^4	3.1412	3.1412	6.28311
10^5	3.1415	3.1415	6.28311
10^6	3.1415	3.1415	6.28317
10^7	3.1415	3.1415	6.28318
10^8	3.1415	3.1415	6.28318

Table 3.11 Case 3
(One end fixed and other attached to weight
$1/\alpha=1000$ and $T_b=\propto$)

T_p	β_i, N=1	β_i, N=2	β_i, N=3
0.01	0.0317	3.1419	6.28334
0.1	0.0331	3.14190	6.28334
1.0	0.0447	3.14190	6.28334
10.0	0.1048	3.14190	6.28334
10^2	0.3177	3.14191	6.28334
10^3	1.0003	3.14191	6.28339
10^4	1.1123	3.14151	6.28308
10^5	1.3112	3.14155	6.28317
10^6	1.4234	3.14158	6.28318
10^8	1.5671	3.14159	6.28318

Fig 3.6. Axial Vibration of Single Bellows Expansion Joint
(One end fixed and other attached to a weight)
(Btai for N=1, 2 & 3 and alpha=0.01)

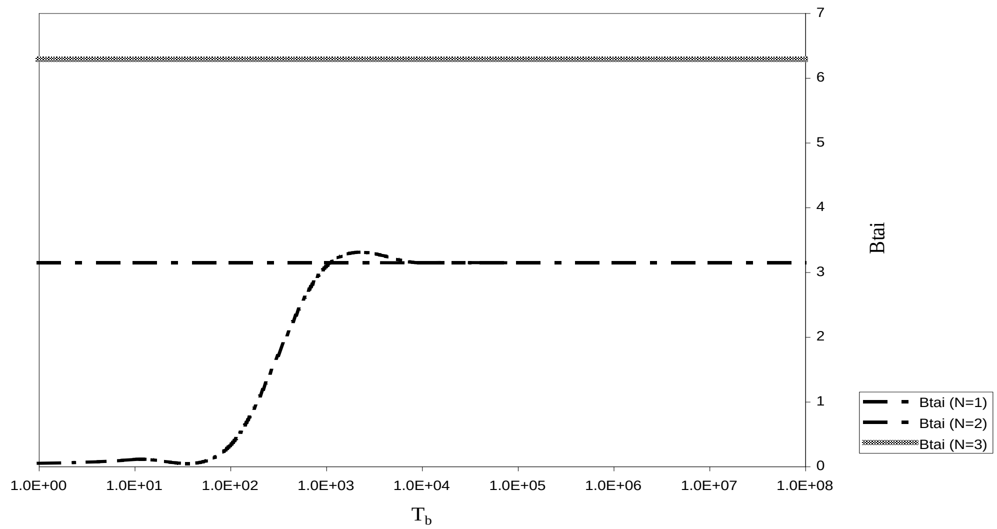
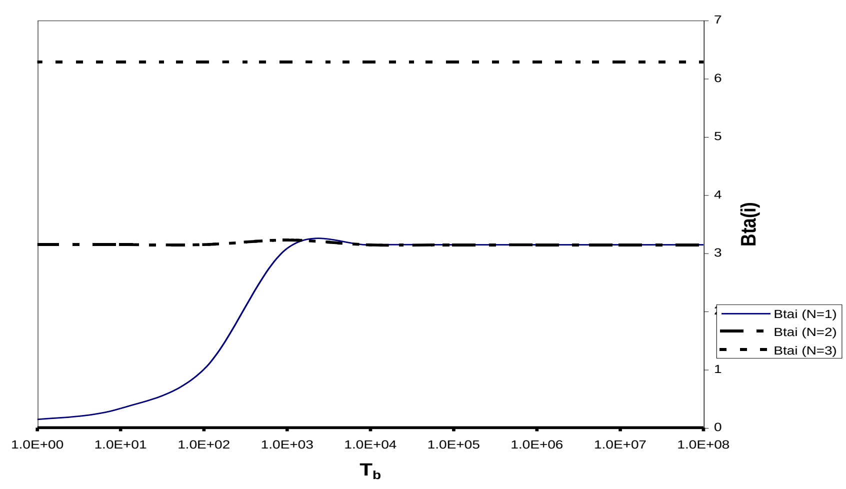


Fig 3.7. Axial Vibrations of Single Bellows Expansion Joint
(One end fixed and other end attached to weight)
(Btai for N=1, 2 & 3 and alpha=0.001)



CHAPTER 4

FINITE ELEMENT ANALYSIS OF AXIAL VIBRATIONS OF SINGLE BELLOWS EXPANSION JOINT RESTRAINED AGAINST ROTATION

4.1 Theoretical Background

The work presented here are the results of investigation of axial vibrations of single bellows expansion joint restrained against rotation on either end using the finite element method. The aim is to model flexible U shaped bellows using bar/rod elements and compare the results obtained by finite element analysis with the theoretical frequencies for the first fundamental mode of vibration.

Rao.C.K & Radhakrishna M [42] analyzed U-shaped bellows subjected to axial force and obtained an analytical solution for the axial vibrations of single bellows that are elastically restrained against rotation on either end using the Euler-Bernoulli beam theory.

As mentioned before the expressions presented either in EJMA code or those derived by investigators, [38] cover classical boundary conditions only. It is observed that the flange-bellow-flange junction - as fixed end conditions are supposed to be infinitely stiff compared to the bellows stiffness and so is not a practical situation compared to the bellows welded on either end in many of the piping systems. Hence, this assumption of fixed-fixed will lead to an over estimation of natural frequencies.

4.2 Characteristics of Bellows

The various geometrical parameters of U-shaped bellows are given in Fig 1.2 r_1 represents the meridional radius of the convolution root, r_2 is the meridional radius of the convolution crown and h is the convolution height. R_m is the mean radius of the bellows - the distance from the bellows centerline to mid convolution height, and t is the bellow material thickness.

It is assumed that $t \ll r_1$, r_2 and $h \ll R_m$ as shown in Fig 1.2. The total length of the bellows is $L = 2(r_1+r_2)$ N. Based on the assumptions given above the bellows are considered to be as an equivalent rod or bar of radius R_m and wall thickness, t.

The other dimensions of the bellows are D_b=322.5mm, depth of convolution, h =24.5mm, pitch, q=22.4mm, number of plies, n=1, overall length of two ends of cylindrical tangent, L=30mm, wall thickness, t = 0.49mm and number of convolutions, N = 9 [2].

4.3 Theory of Free Vibrations of Bellows

The matrix equation for the free vibration of bellows can be written as –

$$[M]\{q\} + [K]\{q\} = 0 \qquad (4.1)$$

Where

{q} -Generalized coordinates

[M] - Mass matrix

[K] - Elastic stiffness matrix

4.4 Formulation of Elastic Stiffness Matrix

The strain energy U of a bellow element of length ‘L’ is given by-

$$U= (½\ AE/L^2)\int_0^1(\partial u/\partial \eta)^2.d\eta + ½\ k.\ L\int_0^1(u)^2.d\eta=0 \tag{4.2}$$

Where-

u is axial deflection, k - axial stiffness, A -area of cross-section and E is the elastic modulus of material of bellows respectively.

Now, assuming a polynomial expression for ‘u’ to be of the form $u = a_1 + b_1 x$– the generalized stiffness matrix for axial vibrations is given by-

$$\int AE.\left(\frac{\partial u}{\partial \eta}\right).d\eta = [K] = \frac{AE}{L}\begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \tag{4.3}$$

4.5 Formulation of Mass Matrix

The mass matrix is found out by using the expression for kinetic energy ‘V’–

$$V= -½\ (\rho Al+W_0)\int(\partial u/\partial t)^2.d\eta \tag{4.4}$$

If G=ρAL, then the above expression becomes-

$$V= ½(G+W_0)\int(\partial u/\partial t)^2.d\eta \tag{4.5}$$

Considering u (η, t) = u (η). e^{ipt} we get, (4.6)

$$V=-\ ½(G+W_0)\ \omega^2\int_0^1 u^2.d\eta \tag{4.7}$$

The generalized mass matrix [M] for axial vibrations is given by-

$$\left(\frac{1}{2}.\rho AL.\right)\int u^2.d\eta = [M] = \frac{\rho AL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \tag{4.8}$$

The matrix equation for free vibration of single bellows is given by-

$$[[K] - \lambda^4 [M]] \{\xi\} = 0 \tag{4.9}$$

Where ξ represent the amplitude of displacement called the mode shape or eigen vector and λ denotes the natural frequency of vibration. It will have a non-zero solution for ξ provided that the determinant of the coefficient matrix

$$\left| [K] - \omega^2 [M] \right| = 0$$

The bellows are modeled using the rod/bar elements -we get the resultant matrix -

$$\begin{bmatrix} 1+R_1-\frac{\lambda^4}{3} & -1+\frac{\lambda^4}{6} & 0 \\ -1+\frac{\lambda^4}{6} & 2-\frac{\lambda^4}{3} & -1+\frac{\lambda^4}{6} \\ 0 & -1+\frac{\lambda^4}{6} & 1+R_2-\frac{\lambda^4}{3} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = 0 \tag{4.10}$$

Where R_1 and R_2 are the stiffness of supports at both ends of the bellows and

$\lambda^4 = (\rho.L^2\omega_n^2) / E$

Now the matrix equation (4.10) can be written as-

$$\begin{bmatrix} (1+R_1) & -1 & 0 \\ -1 & 2 & -1 \\ 0 & 0 & (-1+R_2) \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \frac{\lambda^4}{6} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 4 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \tag{4.11}$$

Since $u_1 \neq 0$ and $u_2 \neq 0, u_3 \neq 0$, the determinant of the matrix is equal to zero-

$$\{(1+R_1-\lambda^4/3)[(2-1/3\lambda^4)(1+R_2-\lambda^4/3)-(-1+\lambda^4/6)^2]\}-(-1+\lambda^4/6)[(-1+\lambda^4/6)(1+R_2-\lambda^4/3)]\}=0 \tag{4.12}$$

Expansion of the determinant equation (4.12) leads to the final frequency equation-

$$\frac{2}{R_1.R_2}\lambda^{12}-9\lambda^8\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{2}{R_1.R_2}\right)+36\lambda^4\left(\frac{2}{R_1}+\frac{2}{R_2}+\frac{1}{R_1.R_2}+1\right)-10^8\left(\frac{1}{R_1}+\frac{1}{R_2}+2\right)=0 \tag{4.13}$$

Where R_1 and R_2 are the rotational restraint parameters and have been defined earlier

It considers two types of elastically restrained end conditions that are as follows –

- Case 1 –One end is fixed and the other end is free
- Case 2 – Both the ends are fixed

The frequencies are obtained for the two cases by considering geometrical dimensions of the bellows mentioned above and results are then compared with those obtained from closed form solutions.

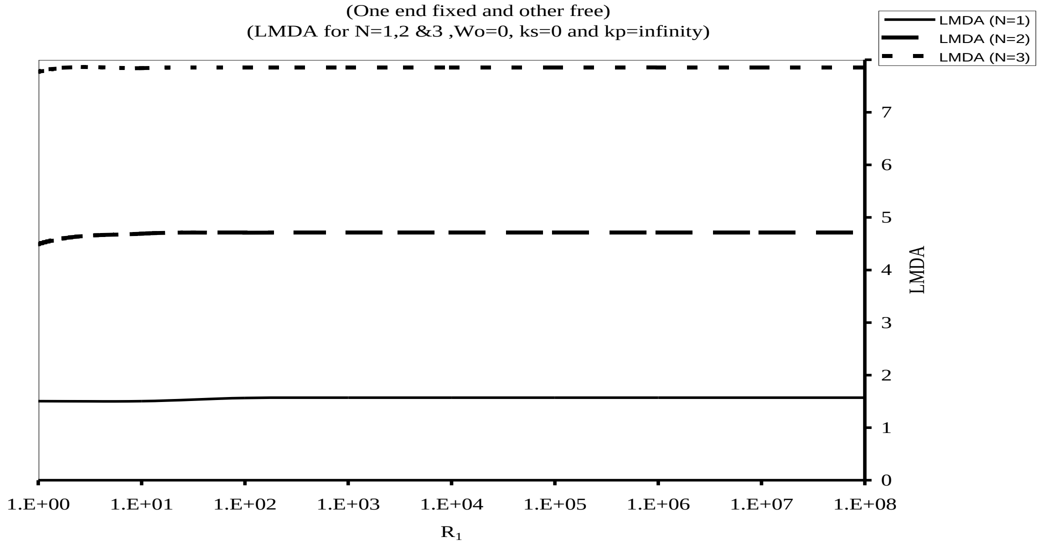
Applying the boundary conditions for both the cases we get λ .

Case 1- represents an end condition of bellows where one end is fixed and other free and both ends elastically restrained i.e. where $R_1=\propto$ or $1/R_1=0$, $W=0$ and $R_2=0$. Substituting the values for R_1 and R_2 in equation (4.13) we get $\lambda_1=1.5507$, $\lambda_2=4.6123$, $\lambda_3=7.6539$ respectively. Also the values for λ are found out for three modes of vibration by increasing the value of R_1 from 0.01 to $\propto$.

Table 4.1 Case 1 (One end fixed and other free)

k_p	**λ_1,N=1**	**λ_2,N=2**	**λ_3,N=3**
0.01	1.4902	3.17331	6.3465
0.1	1.4923	3.3761	6.4862
1.0	1.5022	4.3934	7.5725
10.0	1.5044	4.5910	7.6412
10^2	1.5144	4.61026	7.6527
10^3	1.5146	4.61217	7.6539
10^4	1.5507	4.61236	7.6539
10^5	1.5507	4.61238	7.6539
10^6	1.5507	4.61238	7.6539
10^7	1.5507	4.61238	7.6539
10^8	1.5507	4.61238	7.6539
10^9	1.5507	4.61238	7.6539

Fig 4.1. Axial Vibrations of Single Bellows Expansion Joint
(One end fixed and other free)
(LMDA for N=1,2 &3 ,Wo=0, ks=0 and kp=infinity)

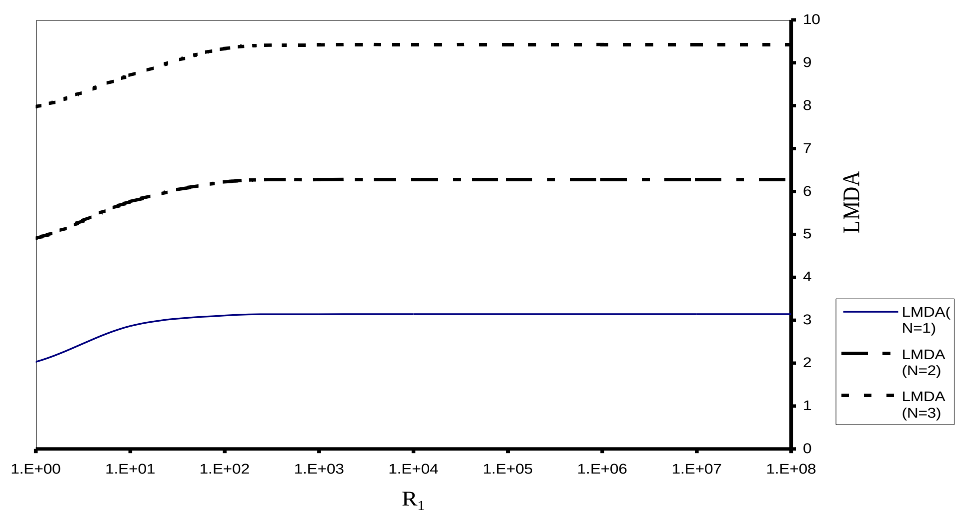


Case 2 –represents a case where both the ends of the bellows are fixed and elastically restrained i.e. where $R_1 = \propto$ and $R_2 = \propto$ or $1/R_1 = 1/R_2 = 0$ & W=0 we get λ_1=3.1316, λ_2=6.28318 and λ_3=9.42477 respectively. By varying R_1 from 0.01 to $\propto$ or 10^9 and keeping R_2 constant the values of λ are found out for N=1,2&3 respectively.

Table 4.2 Case 2 (Both ends fixed)

T_p	λ_1, N=1	λ_2 N=2	λ_3 N=3
0.01	1.57713	4.71451	7.85525
0.1	1.63199	4.72518	7.86669
1.0	2.02875	4.91318	7.97866
10.0	2.86277	5.76055	8.70831
10^2	3.11049	6.22105	9.33172
10^3	3.12845	6.27690	9.41536
10^4	3.13127	6.28257	9.42383
10^5	3.13156	6.28312	9.42468
10^6	3.13158	6.28317	9.42476
10^7	3.13159	6.28318	9.42477
10^8	3.13159	6.28318	9.42477
10^9	3.13159	6.28318	9.42477

Fig 4.2. Axial Vibrations of Single Bellows Expansion Joint
(Both ends fixed)
LMDA for N=1, 2 &3 & R_1 and R_2=infinity)



4.6 Results and Discussion

It is seen that the values of λ obtained using finite element method are found to be in close agreement with the values of β_i obtained from closed form solution [3]. The value of β_i obtained by closed form solution is 1.5707 (N=1) for Case 1 and 3.1415 for Case 2 respectively. The number of elements considered in the present work is two. By increasing the number of elements to 3 it is seen that the values of λ

and β_i would converge and the error percentage in frequency would decrease correspondingly. Figures 4.1 & 4.2 represent the same. Tables 4.3 & 4.4 presents a comparison between frequencies obtained by experiments, exact and finite element methods.

Table 4.3 Comparison of Frequency Solutions in Air (One end fixed and other free (R_2=0 and R_1=∝)

Mode N	**EXP [38] Hz**	**Exact [42] ω, Hz**	**FEA [43], ω Hz**	**Error, %**
1	37.5	40.5	39.99	6.4

Table 4.4 Comparison of Frequency Solutions in Air (Both ends fixed and elastically restrained R_2=∝ and R_1=∝)

Mode N	**EXP[38] Hz**	**Exact [42] ω, Hz**	**FEA [43], ω Hz**	**Error, %**
1	83.0	81.0	80.77	2.6

The theoretical and experimental comparisons confirm that the finite element method developed and used in the study is well adapted to the dynamic response of bellows and gives fairly exact results.

CHAPTER 5

TRANSVERSE VIBRATIONS OF SINGLE BELLOWS EXPANSION JOINT RESTRAINED AGAINST ROTATION

This chapter deals with the theoretical investigation of transverse vibrations of single bellows subjected to internal pressure, rotatory inertia and considering the ends as elastically restrained against rotation. The mathematical model represents the expansion joint with the ends restrained against rotation as shown in Fig 5.1. A non-dimensional rotational restraint parameter is defined in order to derive a closed form general-purpose transcendental frequency equation.

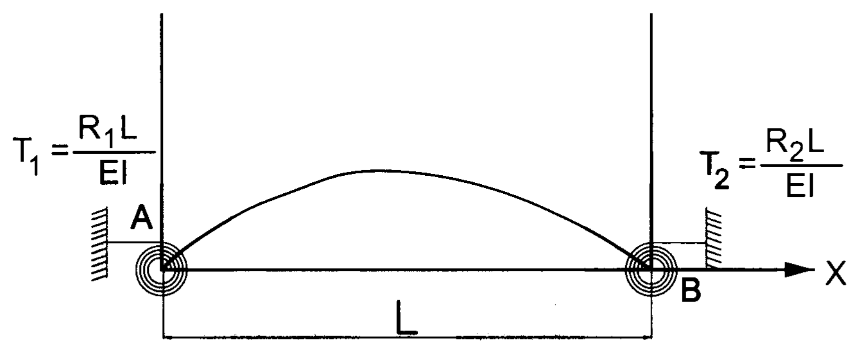


Fig 5.1 Mathematical Model of Single Bellows Expansion Joint (Lateral Mode)

5.1 The Differential Equation

The general form of the differential equation of vibration of bellows for single or Universal double bellows expansion joint is given by –

$$EI\frac{\partial^4 w}{\partial x^4} + P\pi R_m{}^2\frac{\partial^2 w}{\partial x^2} - J\frac{\partial^4 w}{\partial x^2 \partial t^2} + m_{tot}\frac{\partial^2 w}{\partial t^2} = 0 \tag{5.1}$$

Where EI is bending stiffness, P-internal pressure, J- total mass moment of inertia per unit length, w – deflection, x-axial coordinate, R_m – mean radius of bellows, m_{tot} is the total mass of bellows per unit length includes mass of bellows and fluid mass and t-time [28]. Using the technique of Separation of variables, the transverse deflection of the bellows axis 'w' can be expressed as –

$$w(x, t) = X(x) . T(t) \tag{5.2}$$

Where X (t) is a parameter of x only and T (t) is the time harmonic function (T is associated with time here) such as –

$$T(t) = A\,e^{i\omega t} \tag{5.3}$$

Differentiating the above equation (5.2) and substituting in the differential equation (5.1) we get –

$$\frac{\partial^4 X}{\partial x^4} + \frac{\left(P\pi R_m{}^2 + J.w^2\right)}{EI}\frac{\partial^2 X}{\partial x^2} - w^2\frac{m_{tot}}{EI}X = 0 \tag{5.4}$$

$$\text{If } c = \sqrt{\frac{\left(P\pi R_m{}^2 + J\omega^2\right)}{2EI}} \tag{5.5}$$

$$\frac{d^4X}{dx^4} + 2c^2\frac{d^2X}{dx^2} - \lambda^4 X = 0 \tag{5.6}$$

The general solution of the equation is given by –

$$X(x) = A \sinh \alpha x + B \cosh \alpha x + C \sin\beta x + D \cos\beta x \qquad (5.7)$$

The first three derivatives of equation (5.7) are given as follows-

$$\frac{dX(x)}{dx} = A\alpha \cosh \alpha x + B\alpha \sinh \alpha x + C\beta \cos \beta x + D\beta \sin \beta x \qquad (5.8)$$

$$\frac{dX(x)^2}{dX^2} = A\alpha^2 \sinh \alpha x + B\alpha^2 \cosh \alpha x - C\beta^2 \sin \beta x - D\beta^2 \cos \beta x \qquad (5.9)$$

Where the roots of the equation are α & β and their values are given by –

$$\alpha = \sqrt{-c^2 + \sqrt{c^4 + \lambda^4}} \qquad (5.10)$$

$$\beta = \sqrt{c^2 + \sqrt{c^4 + \lambda^4}} \qquad (5.11)$$

$$\lambda^4 = \sqrt[4]{\frac{m_{tot.}\omega^2}{EI}} \qquad (5.12)$$

Where A, B, C, D are arbitrary constants

5.2 Derivation of the Frequency Equation for Transverse Vibrations of Elastically Restrained Single Bellows Expansion Joint

At end 'A' the bellows are connected to a pipe nipple and is considered to have a rotational stiffness of R_1 and R_2 at either end as shown in figure 5.1.

At x = 0 the boundary conditions are given by-

X (0) = 0 and

$$EI\,\frac{\partial^2 X(O)}{\partial x^2} = -T_1\,\frac{\partial X(O)}{\partial x} \tag{5.13}$$

At end B, the boundary conditions are given by –

X = L
X (L) = 0 and (5.14)

$$\frac{\partial^2 X(L)}{\partial x^2} = -T_2\,\frac{\partial X(L)}{\partial x} \tag{5.15}$$

$T_1 = R_1\, L/EI$ (5.16)

$T_2 = R_2\, L/EI$ (5.17)

Or can also be written as-

$$X(0) = 0 \qquad EI\,\frac{d^2x(0)}{dx^2} = R_1\,\frac{dx(0)}{dx};$$

$$X(L) = 0 \qquad EI\,\frac{d^2x(L)}{dx^2} = R_2\,\frac{dx(L)}{dx};$$

Applying the boundary conditions we get-

$$B + D = 0 \tag{5.18}$$

$$EI\,[B(\alpha L)^2 - D(\beta L)^2] = T_1\,[A(\alpha L) + C(\beta L)] \tag{5.19}$$

$$\{B\,(\alpha L)^2 - D(\beta L)^2\} = \frac{R_1 L}{EI}\{A(\alpha L) + C(\beta L)\} \tag{5.20}$$

$$A\,\text{Sinh}\alpha L + B\cosh\alpha L + C\sin\beta L + D\cos\beta L = 0 \tag{5.21}$$

$A\{((\alpha L)^2 \sinh\alpha L + T_2(\alpha L)\cosh\alpha L)\} + B\{(\alpha L)^2\cosh\alpha L + T_2\,(\alpha L)\sinh\alpha L)\} +$

$C\{(T_2\beta L$

$\cos\beta L - (\beta L^2)\sin\beta L)\} - D\{((\beta L)^2\cosh\beta L + T_2\,\beta L\sin\beta L)\} = 0$ (5.22)

Re-writing the above equation by re-arranging the terms we get-

$\{A\,(\alpha L)^2 \sin h\alpha L + B(\alpha L)^2\cosh\alpha L + C(\beta L)^2\sin\beta L + D(\beta L)^2\cosh\beta L\} = -\left(\frac{R_2 L}{EI}\right)$

$\{A\alpha L\cosh\alpha L + B\alpha L\sinh\alpha L + C\beta L\cosh\beta L - \text{Sin}\beta L\}$ (5.23)

$A\,\{(\alpha L)\,(\alpha L\sinh\alpha L + T_2\cosh\alpha L\} + B\,(\alpha L)\,\{(\alpha L\cosh\alpha L + T_2\sinh\alpha L]$

$+ C\,(\beta L)\,\{T_2\cosh\beta L - (\beta l)\sin\beta L\} - D\,(\beta L)\,\{(T_2\sin\beta L + (\beta L)\cos\beta L\} = 0$ (5.24)

Writing the above set of equations in matrix form, we get-

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ T_1(\alpha L) & -\alpha L^2 & T_1(\beta L) & \beta L^2 \\ \sinh\alpha L & \cosh\alpha L & \sin\beta L & \cos\beta L \\ c_1 & c_2 & c_3 & -c_4 \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \\ D \end{Bmatrix} = 0 \tag{5.25}$$

Where, $c_1 = (\alpha L)\,[(\alpha L)\sinh\alpha L + T_2\cosh\alpha L]$

$c_2 = (\alpha L)\,[(\alpha L\cosh\alpha L + T_2\sinh\alpha L]$

$c_3 = (\beta L)\,[T_2\cos\beta L - (\beta L)\sin\beta L]$

$c_4 = (\beta L)\,[T_2\sin\beta L + (\beta L)\cos\beta L]$

(5.26)

$$\begin{bmatrix} T_1(\alpha l) & T_1(\beta l) & +(\beta l)^2 \\ \text{Sinh}\,(\alpha l) & \text{Sin}\,(\beta l) & \text{Cos}(\beta l) \\ c_1 & c_3 & -c_4 \end{bmatrix} + \begin{bmatrix} T_1(\alpha l) & -(\alpha l)^2 & T_1(\beta l) \\ \text{Sinh}\,(\alpha l) & \text{Cosh}(\alpha l) & \text{Sin}\,(\beta l) \\ c_1 & c_2 & c_3 \end{bmatrix} = 0 \tag{5.27}$$

Expanding the above equation –we get

$$T_1(\alpha L)\,[-c_4 \sin(\beta L) - c_3 \cos(\beta L)] - T_1(\beta L)[-c_4 \sinh(\alpha L) - c_1 \cos(\beta L)] + (\beta L)^2\,[c_3 \sinh(\alpha L) - c_1 \sin(\beta L)] + T_1(\alpha L)\,[c_3 \cosh(\alpha L) - c_2 \sin(\beta L)] + (\alpha L)^2[c_3 \sinh(\alpha L) - c_1 \sin(\beta L)] + T_1(\beta L)\,[c_2 \sinh(\alpha L) - c_1 \cosh(\alpha L)] = 0 \tag{5.28}$$

Substituting c_1, c_2, c_3 & c_4 in above equation and simplifying we get-

$$-c_4 \sin\beta L - c_3 \cos\beta L = -(\beta L) \sin\beta L\,[T_2 \sin\beta L + (\beta L) \cos\beta L] - (\beta L) \cos\beta L\,[T_2 \cos\beta L - (\beta L)\sin\beta L] \tag{5.29}$$

$$= -(\beta L)\, T_2 \sin^2\beta L - (\beta L)^2 \cos\beta L \sin\beta L - (\beta L)\, T_2 \cos^2\beta L + (\beta L)^2 \sin\beta L \cos\beta L$$

$$\therefore [c_4 \sin\beta L - c_3 \cos\beta L] = -T_2(\beta L)\,(\sin^2\beta L + \cos^2\beta L) = -T_2(\beta L) \tag{5.30}$$

Now,

$$-c_4 \sinh(\beta L) - c_1 \cos\beta L) = -(\beta L) \sinh\alpha L\,[T_2 \sin\beta L + \beta L\cos\beta L] - (\alpha L) \cos\beta L\,[(\alpha L) \sinh\alpha L + T_2 \cosh\alpha L] \tag{5.31}$$

$$= -T_2\,(\beta L) \sinh\alpha L \sin\beta L - (\beta L)^2 \sinh\alpha L \cos\beta L - (\alpha L)^2 \sinh\alpha L\cos\beta L - T_2(\alpha L)\cosh\alpha L\cos\beta L \tag{5.32}$$

$$= -T_2\,[(\alpha L \cosh\alpha L \cos\beta L + (\beta L) \sinh\alpha L \sin\beta L] - [(\alpha L)^2 + (\beta L)^2] \sinh\alpha L \cos\beta L$$

c_4 sinhαL – $c_1$cosβL = – T_2 [(αL) coshαL cosβL + (βL) sinhαL sinβL] – [(αL)^2 + (βL)^2] sinhαL cosβL] (5.33)

Now,

[c_3 sinh(αL) –c_1 sin(βL)] = (βL) sinh(αL) [T_2 cosβL –(βL) sinβL – (αL) sinβL [αL sinhαL + T_2 coshαL] = T_2 [(αL) coshαL sinβL + (βL) sinhαL cosβL] – [(αL)^2 + (βL)^2 sinhαL sinβL (5.34)

c_3 sinh(αL) – c_1 sin(βL) = {T_2 [(αL) coshαLsinβL+βLsinhαLcosβL] – [(αL)^2+(βL)^2]sinhαLsinβL} (αL)^2 (5.35)

c_3 cosh(αL) – c_2 sin(βL) = βL coshαL [T_2 cosβL-βL sinβL] -αL sinβL [αL coshαL+T_2 sinhαL] (5.36)

c_3 cosh(αL) – c_2 sin(βL) = {T_2 [(βL) coshαl cosβl - (αL) sinhαL sinβL] – [(αl)^2 + (βl)^2] coshαL sinβL}. T_1(αL) (5.37)

c_2 sinhαL – c_1 coshαL = (αL) sinhαl [αLcoshαL +T_2sinhαL] –αL[coshαL[αLsinhαL+T_2coshαL] (5.38)

Collecting and rearranging all the terms we determine frequency equation as follows-

– T_1 T_2 (αL) (βL) + T_1 T_2 (βL) [(αL) coshαL cosβL + (βL) sinhαL sinβL] + T_1 (βL) [(αL)^2 + (βL)^2] sinhαL cosβL– T_2 [(αL)^2 + (βL)^2] [(αL) coshαL sinαL – (βL)

sinhαL cosβL] – [$(\alpha L)^2 + (\beta L)^2]^2$ sinhαL sinβL –$T_2[(\alpha L)^2+(\beta L)^2]$.[$\alpha$Lcosh$\alpha$Lsin$\beta$L-$\beta$Lsinh$\alpha$Lcos$\beta$L]-$[(\alpha L)^2+(\beta L)^2]$. $[(\alpha L)^2+(\beta L)^2]$ sinhαL sinβL-T1.T2 (αL.βL)

(5.39)

= {– 2(αL) (βL) +(αL) (βL) coshαL coshβL +$(\beta L)^2$ sinhαL sinβL +(αL) (βL) coshαL cosβL$(\alpha L)^2$ sinhαL sinβL] T_1T_2 + T_1 {(βL) $[(\alpha L)^2 + (\beta L)^2]$ SinhαL cosβL –(αL)$[(\alpha L)^2 + (\beta L)^2]$ coshαl sinβL} –$T_2[(\alpha L)^2 + (\beta L)^2]$ [(αL) coshαL sinβL – (βL) sinhαL cosβL] – $[(\alpha L)^2 + (\beta L)^2]$ sinhαL sinβL = 0 (5.40)

–T_1 T_2 {$[(\alpha L)^2(\beta L)^2]$ sinhαL sinβL + 2 (αL) (βL) (1–coshαL cosβL)} + T_1 $[(\alpha L)^2 + (\beta L)^2]$ [αL coshαL sinβL –(βL) sinhαL cosβL] + T_2 $[(\alpha L)^2 + (\beta L)^2]$ [αL coshαL sinβL – (βL) sinhαL cosβL] + $[(\alpha L)^2 + (\beta L)^2]^2$ sinhαL SinβL = 0 (5.41)

$\therefore$ {$[(\alpha L)^2 + (\beta L)^2]^2 + [(\alpha L)^2 - (\beta L)^2[$ T_1T_2} sinhαL sinβL + 2(αL) (βL) T_1T_2 (1-coshαL cosβL) = 0 (5.42)

The final frequency equation is given by-

$$\{[(\alpha L)^2 - (\beta L)^2] + \frac{1}{T_1T_2}[(\alpha L)^2 + (\beta L)^2]\} \sinh\alpha L \sin\beta L + 2\alpha L.\ \beta L\ (1-\cosh\alpha L \cos\beta L)$$

$$+ \left(\frac{1}{T_2} + \frac{1}{T_1}\right) [(\alpha L)^2 + (\beta L)^2\ [\ (\alpha L) \cosh\alpha L \sin\beta L\ - (\beta L) \sinh\alpha L \cos\beta L] = 0 \quad (5.43)$$

If T_1 = T_2 = $\propto$ we get

$[(\alpha L)^2 - (\beta L)^2]$ sinhαL sinβL + (βL) (1– coshαl cosβL) = 0

$$[(\beta L)^2 - (\alpha L)^2] \sinh(\alpha L) \sinh(\beta L) + 2(\alpha L) (\beta L) [\cosh(\alpha L) \cos (\beta L) - 1] = 0 \quad (5.44)$$

5.3 Derivation of the Mode Shape Expression for Transverse Vibrations of Elastically Restrained Single Bellows Expansion Joint

We know

$$y = A\sinh(\alpha L)\xi + B\cosh(\alpha L)\xi + C\sin(\beta L)\xi + D\cos(\beta L)\xi$$

(5.45)

$$\xi = \frac{x}{L}$$

Where

The boundary conditions for a single bellows are given by-

$y = 0;$ at $\xi = 0$

$$\frac{\partial^2 y}{\partial \xi^2} = -T_1 \frac{\partial y}{\partial \xi} \text{ at } \xi = 0 \rightarrow T_1 = \left(\frac{R_1 L}{EI}\right)$$

$y = 0;$ at $\xi = L$

$$\frac{\partial^2 y}{\partial \xi^2} = T_2 \frac{\partial y}{\partial \xi} \text{ at } \xi = L \rightarrow T_2 = \left(\frac{R_2 L}{EI}\right)$$

$$\frac{\partial y}{\partial \xi} = (\alpha L)[A\cosh(\alpha L)\xi + B\mathrm{Sinh}(\alpha L)\xi] + (\beta L)[C\cos(\beta L)\xi - D\mathrm{Sin}(\beta L)\xi] \quad (5.46)$$

$$\frac{\partial^2 y}{\partial \xi^2} = (\alpha L)^2 [A\mathrm{Sinh}(\alpha L)\xi + B\mathrm{Cosh}(\alpha L)\xi] - (\beta L)^2 [C\mathrm{Sin}(\beta L)\xi + D\mathrm{Sin}(\beta L)\xi] \quad (5.47)$$

At $\xi = 0;\ y = 0;\ B + D = 0;\ (\alpha L)^2 [B] - (\beta L)^2 [D] = -T_1 [(\alpha L)A + (\beta L)C]$

At $\xi = 1; (\alpha L)^2 [A\sinh(\alpha L) + B\cosh(\alpha L)] - (\beta L)^2 [C\sin\beta L + D\cos\beta L] +$

$$= T_2 \{(\alpha L)[A\cosh(\alpha L) + B\sinh(\alpha L)] + (\beta L)[C\cos(\beta L) - D\mathrm{Sin}(\beta L)]\} \quad (5.49)$$

$$B + D = 0$$
$$\therefore D = -B$$

$$B(\alpha L)^2 - D(\beta L)^2 = T_1 \left[A(\alpha L)A + (\beta L)C\right] \tag{5.50}$$

$$A\sinh(\alpha L) + B\cosh(\alpha L) + C\sin(\beta L) + D\cos(\beta L) = 0 \tag{5.51}$$

$$\text{At } \xi = 1; (\alpha L)^2 \left[A\sinh(\alpha L) + B\cosh(\alpha L)\right] - (\beta L)^2 \left[C\sin\beta L + D\cos\beta L\right]$$

$$= T_2 \left\{(\alpha L)\left[A\cosh(\alpha L) + B\sinh(\alpha L)\right] + (\beta L)\left[C\cos(\beta L) - D\sin(\beta L)\right]\right\} \tag{5.52}$$

$$\frac{B}{D} = -1 \tag{5.53}$$

$$[(\alpha L)]A + [(\beta L)]C = +\frac{D}{T_1}\left[(\alpha L)^2 + (\beta L)^2\right] \tag{5.54}$$

$$[\sinh(\alpha L)]A + [\sin\beta L]C = -D[\cos\beta L + \cosh\alpha L] \tag{5.55}$$

$$\left\{\left[(\alpha L)^2 \sinh(\alpha L)\right]A - \left[(\beta L)^2 \sin\beta L\right]C - \left[T_2(\alpha L)\cosh\alpha L\right]A - \left[T_2(\beta L)\cos\beta L\right]C\right\}$$

$$= \left\{T_2(\alpha L)[\sinh(\alpha L)]B - [T_2(\beta L)\sin\beta L]D - B\left|(\alpha L)^2 \cosh\alpha L\right| + D\left|(\beta L)^2 \cos\beta L\right|\right\} \tag{5.56}$$

$$\begin{gathered} A\{(\alpha L)[(\alpha L)\sinh\alpha L - T_2\cosh\alpha L]\} - C\{(\beta L)[(\beta L)\sin\beta L + T_2\cos\beta L]\} \\ = D\left\{-T_2(\alpha L)\sinh\alpha L - T_2(\beta L)\sin\beta L + (\beta L)^2\cos\beta L + (\alpha L)^2\cosh\alpha L\right\} \end{gathered} \tag{5.57}$$

$$\begin{gathered} A(\alpha L)[(\alpha L)\sinh\alpha L - T_2\cosh\alpha L] - C(\beta L)[(\beta L)\sin\beta L + T_2\cos\beta L] \\ = D\left\{(\alpha L)^2\cosh\alpha L + (\beta L)^2\cos\beta L - T_2[\alpha L\sinh\alpha L + \beta L\sin\beta L]\right\} \end{gathered} \tag{5.58}$$

$$\frac{B}{D} = -1$$

$$(\alpha L)A + (\beta L)C = \frac{D}{T_1}\left[(\alpha L)^2 + (\beta L)^2\right] \tag{5.59}$$

$$[\sinh \alpha L]A + [\sin \beta L]C = -D[\cos\beta L + \cosh \alpha L] \tag{5.60}$$

Multiplying (5.59) by sinβL and (5.60) by βL and canceling the like terms we get,

$$5.59 x \sin \beta L : [(\alpha L)\sin \beta L]A + [(\beta L)\sin \beta L]C = \frac{D \sin \beta L}{T_1}\left[(\alpha L)^2 + (\beta L)^2\right]$$

$$5.60x\ \beta L : [(\beta L)\sin \alpha L]A \mp [(\beta L)\sin \beta L]C = \pm D(\beta L)\ \ [\cosh \alpha L + \cos \beta L]$$

On subtracting, we get-

$$A[\alpha L \sin \beta L - \beta L \sinh \alpha L] = D\left\{\frac{\sin \beta l}{T_1}\left[(\alpha l)^2 + (\beta L)^2\right] + (\beta L)[\cosh \alpha L + \cosh \beta L]\right\} \tag{5.61}$$

$$A' = \frac{A}{D} = \frac{\left[(\alpha L)^2 + (\beta L)^2\right]\sin \beta L + T_1(\beta L)[\cosh \alpha l + \cos \beta L]}{T_1[\alpha L \sin \beta L - \beta L \sinh \alpha L]} \tag{5.62}$$

To find the ratio C'=C/D, we multiply (5.59) by sin hαL and (5.60) by αL -

$$(5.59)\, x \sinh \alpha L : [(\alpha L)\sinh \alpha L]A + [\beta L \sinh \alpha L]C = \frac{D \sinh \alpha L}{T_1}\left[(\alpha L)^2 + (\beta L)^2\right]$$

$$(5.60)\, x\ \ \alpha L : [(\alpha L)\sinh \alpha L]A + [\alpha L \sinh \beta L]C = -D(\alpha L)[\cosh \alpha L + \cos \beta L]$$

Subtracting we get-

$$-C\left[(\alpha L)\sin\beta L-(\beta L)\sinh\alpha L\right]$$
$$=D\left\{\frac{\left[(\alpha L)^2+(\beta L)^2\right]}{T_1}\sinh\alpha L+(\alpha L)\left[\cosh\alpha L+\cos\beta L\right]\right\} \tag{5.63}$$

$$-\frac{C}{D}=\left\{\frac{\left[(\alpha L)^2+(\beta L)^2\right]\sinh\alpha L+T_1(\alpha L)\left[\cosh\alpha L+\cos\beta L\right]}{T_1\left[\alpha L\sin\beta L-\beta L\sinh\alpha L\right]}\right\} \tag{5.64}$$

$$C'=\frac{C}{D}=-\frac{\left[(\alpha L)^2+(\beta L)^2\right]\sinh\alpha L+T_1(\alpha L)\left[\cosh\alpha L+\cos\beta L\right]}{T_1\left[\alpha L\sin\beta L-\beta L\sinh\alpha L\right]} \tag{5.65}$$

$$\frac{B}{D}=-1$$

$$A'=\frac{A}{D}=-\frac{\left[(\alpha L)^2+(\beta L)^2\right]\sinh\beta L+T_1(\beta L)\left[\cosh\alpha L+\cos\beta L\right]}{T_1\left[\alpha L\sin\beta L-\beta l\sinh\alpha L\right]}$$

$$\therefore\ y(\xi)=D\left\{\left[\cos\beta L\,\xi-\cosh(\alpha L)\xi\right]+\left(\frac{C}{D}\right)\sin\beta L\xi+\left(\frac{A}{D}\right)\sinh\alpha L\,\xi\right\}$$

$$\left\{\frac{y(\xi)}{D}\right\}=\left\{\begin{array}{l}\left[\cos\beta L\,\xi-\cosh\alpha L\,\xi\right]+\left\{\dfrac{\left[(\alpha L)^2+(\beta L)^2\right]}{T_1\left[\alpha L\sin\beta L-\beta L\sinh\alpha L\right]}\right\}\sin\beta L\xi-\\ \left\{\dfrac{\left[(\alpha L)^2+(\beta L)^2\right]\sinh\alpha L+T_1(\alpha L)\left[\cosh\alpha L+\cos\beta L\right]}{T_1\left[\alpha L\sin\beta L-\beta L\sinh\alpha L\right]}\right\}\sinh\alpha L\,\xi\end{array}\right\}$$

$$(\alpha L)A+(\beta L)C=\frac{D}{T_1}\left[(\alpha L)^2+(\beta L)^2\right]$$

$$A(\alpha L)\left[(\alpha L)\sinh\alpha L-T_2\cosh\alpha L\right]-C(\beta L)\left[(\beta L)\sin\beta L+T_2\cos\beta L\right]$$

$$=D\left\{\left[(\alpha L)^2\cosh\alpha L+(\beta L)^2\cos\beta L\right]-T_2\left[\alpha L\sinh\alpha L+\beta L\sin\beta L\right]\right\}$$

$$(2)\ \mathrm{x}\ \left[\alpha L\sinh\alpha L-T_2\cosh\alpha L\right]$$

$$A(\alpha L)\left[\alpha L\sinh\alpha L-T_2\cosh\alpha L\right]+(\beta L)C\left[(\alpha L)\sin\alpha L+T_2\cos\alpha L\right]$$

$$=\frac{D}{T_1}\left[(\alpha L)^2+(\beta L)^2\right]\left[\alpha L\ \sinh\ \alpha L-T_2\cosh\alpha L\right]$$

$$(\beta L)C\left[\alpha L\ \sinh\ \alpha L-T_2\cosh\alpha L\right]+(\beta L)C\left[\beta L\ \sinh\ \beta L-T_2\cosh\beta L\right]$$
$$(\beta L)$$

$$=\frac{D}{T_1}\left[(\alpha L)^2+(\beta L)^2\right]\left[\alpha L\ \sinh\ \alpha L-T_2\cosh\alpha L\right]$$

$$-D\left\{\left[(\alpha L)^2\cosh\alpha L+(\beta L)^2\cosh\beta L\right]-T_2\left[\alpha L\sinh\ \alpha L+\beta L\sin\ \beta L\right]\right\}$$

$$T_1\left[\frac{C}{D}\right](\beta L)\left\{\left[\alpha L\sinh\ \alpha L+\beta L\sin\ \beta L\right]+T_2\left[\cos\beta L-\cosh\alpha L\right]\right\}$$

$$=\left[(\alpha L)^2+(\beta L)^2\right]\left[\alpha L\ \sinh\ \alpha L-T_2\cosh\alpha L\right]$$

$$-T_1\left\{\left[(\alpha L)^2\cosh\alpha L+(\beta L)^2\cosh\beta L\right]-T_2\left[\alpha L\sinh\ \alpha L+\beta L\sin\ \beta L\right]\right\}$$

$$C'=\frac{C}{D}=\left\{\frac{\left[(\alpha L)^2+(\beta L)^2\right]\left[\alpha L\ \sinh\ \alpha L-T_2\cosh\alpha L\right]-T_1\left[(\alpha L)^2\cosh\alpha L+(\beta L)^2\cosh\beta L\right]+T_1T_2\left[\alpha L\sinh\ \alpha L+\beta L\sin\ \beta L\right]}{T_1(\beta L)\left[\alpha L\sinh\ \alpha L+\beta L\sin\ \beta L\right]+T_1T_2\left(\cos\beta L-\cosh\alpha L\right)}\right\}$$

$$\left(\frac{C}{D}\right)=\frac{\left\{\left[(\alpha L)^2+(\beta L)^2\right](\alpha L)\sinh\ \alpha L-T_2\left[(\alpha L)^2+(\beta L)^2\right]\cosh\ \alpha L-T_1\left[\begin{array}{l}(\alpha L)^2\cosh\alpha L+(\beta L)^2\cos\beta L+\\ T_1T_2\left[\alpha L\sinh\ \alpha L+\beta L\sin\ \beta L\right]\end{array}\right]\right\}}{\left\{T_1(\beta L)\left[\alpha L\sinh\ \alpha L+\beta L\sin\ \beta L\right]+T_1T_2\left[\cos\beta L-\cosh\alpha L\right]\right\}}$$

$2\ x\ \left[\beta L\ \sin\beta L+T_2\cos\beta L\right]$:

$$(\alpha L)A\left[\beta L\ \sin\ \beta L+T_2\cos\beta L\right]+(\beta L)C\left[\beta L\ \sin\ \beta L+T_2\cos\beta L\right]$$

$$=\frac{D}{T_1}\left[(\alpha L)^2+(\beta L)^2\right]\left[\beta L\ \sin\ \beta L+T_2\cos\beta L\right]$$

$$(\alpha L)A\left[\alpha L\ \sinh\ \alpha L-T_2\cosh\alpha L\right]-(\beta L)c\left[\beta L\ \sin\ \beta L+T_2\cos\beta L\right]$$

$$= D\{[(\alpha L)^2 \cosh\alpha L + (\beta L)^2 \cos\beta L] - T_2[\alpha L \sinh\alpha L + \beta L \sin\beta L]\}$$

Adding-

$$(\alpha L)A\{[\alpha L \sinh\alpha L + \beta L \sin\beta L] + T_2[\cos\beta L - \cosh\alpha L]\}$$

$$\begin{aligned} &= \frac{D}{T_1}\left[(\alpha L)^2 + (\beta L)^2\right][\beta L \sin\beta L + T_2 \cos\beta L] \\ &+ D\{[(\alpha L)^2 \cosh\alpha L + (\beta L)^2 \cos\beta L] - T_2[\alpha L \sinh\alpha L + \beta L \sin\beta L]\} \end{aligned}$$

$$\left(\frac{A}{D}\right)\{T_1(\alpha L)[\alpha L \sinh\alpha L + \beta L \sin\beta L] + T_1T_2(\alpha L)[\cos\beta L - \cosh\alpha L]\}$$

$$= \left\{\begin{array}{l} \left[(\alpha L)^2 + (\beta L)^2\right][\beta L \sin\beta L + T_2 \cos\beta L] + T_1\left[(\alpha L)^2 \cosh\alpha L + (\beta L)^2 \cos\beta L\right] \\ - T_1T_2[\alpha L \sinh\alpha L + \beta L \sin\beta L] \end{array}\right\}$$

$$A' = \frac{A}{D} = \frac{\left\{\begin{array}{l} \left[(\alpha L)^2 + (\beta L)^2\right][\beta L \sin\beta L + T_2 \cos\beta L] + T_1\left[(\alpha L)^2 \cosh\alpha L + (\beta L)^2 \cos\beta L\right] - \\ T_1T_2[\alpha L \sinh\alpha L + \beta L \sin\beta L] \end{array}\right\}}{T_1(\alpha L)\{[\alpha L \sinh\alpha L + \beta L \sin\beta L] + T_2[\cos\beta L - \cosh\alpha L]\}}$$

Mode shape can be therefore be written as-

$$\left[\frac{y(\xi)}{D}\right] = A' \sinh(\alpha L)\xi - \cosh(\alpha L)\xi + C' \sin(\beta L)\xi + \cos(\beta L)\xi$$

Or can be written as-

$$\begin{aligned} &\left[\frac{y(\xi)}{D}\right] \\ &= [C' \sin(\beta L)\xi + A' \sin(\alpha L)\xi] + [\cos(\beta L)\xi - \cos(h\alpha L)\xi] \end{aligned}$$

5.4 Results and Discussion

Equation (5.44) is a closed-form general-purpose transcendental frequency equation for finding out the natural frequencies of transverse vibrations of single bellows expansion joints restrained against rotation on either ends and defined by non-dimensional rotational restraint parameters such as $T_1$and T_2.

By providing different values of non-dimensional parameters 'c' which depends on internal pressure 'P', bellows moment of inertia 'J' and T_1 & T_2 that depend on rotational spring stiffness R_1 and R_2, the transcendental equation can be solved for values of 'λ' for various modes of vibration 'N'.

The bisection method of trial and error is utilized in this study in obtaining various values of frequency parameter 'λ' for a given value of c, T_1 and T_2.

To show the effect of rotational restraint parameters T_1 and T_2 on frequencies of vibration of bellows, a single bellow with length L = 0.0693m, mass moment of inertia per unit length, J = 0.001153kgm, flexural stiffness EI = 5.078Nm and total mass m_{tot} = 5.138kg/m is considered. The above dimensions of bellows are same as those considered by Jakubauskas.V.F.

For $T=T_1=T_2=0$, the general frequency equation (5.44) reduces to a case of single bellows with simply supported ends. Similarly, for $T_1=T_2=\propto$, the equation reduces to a single bellows with fully fixed and elastically restrained end condition.

The frequency equation derived here is exact and can be applied to all types of single bellows with unequal rotational restraints on either end.

According to [1], the maximum internal pressure in bellows is given by –

(5.45)

$EI = 0.25\ k.P.R_m^2$ (5.46)

Substituting (5.45) and (5.46) and the dimensional and material property data of the bellows expansion joint given above into expressions (5.6) and (5.7), we get –

$$c = \sqrt{\left(616.49 + 0.0001135\omega^2\right)} \tag{5.47}$$

$$\lambda = 1.0029\sqrt{\omega} \tag{5.48}$$

In order to study the effect of variation of internal pressure on the lateral natural frequency is investigated. The first four modes of transverse natural frequencies are obtained for internal pressures of P=0.0MPa and P=166.0MPa and are given in Tables 5.1 & 5.2. The same are graphically represented in figures 5.2 & 5.3 respectively. The effect of rotational restraint on the natural frequency is studied for an initial value of 0.01 to a maximum of 10^{10} and is assumed to be equal on either end.

Firstly, it is seen that as internal pressure of the bellows increases the transverse natural frequency decrease. For example at P=0.0MPa and mode of vibration N=1, natural frequency obtained is ω=1690.477 radians/s and at

P=166.0MPa and N=1, and ω=1069.4297 radians/s a drastic drop by about 37%. The same is true for even higher order of modes of vibration N=2, 3 & 4.

It is also seen that as the value of rotational restraint T – increases from 0.01 to 10^{10}, the frequencies tend to increase for all modes of vibration. As the rotational stiffness value $T \rightarrow \propto$, the frequency value increase by about 54% for N=1 at P=0.0MPa and by 69% for N=1 at P=166.0MPa respectively. However, it is observed that there is no change in frequency and it becomes constant from T=10^4 onwards.

Table 5.1 gives a comparison of the exact frequency solutions obtained using the bisection method for a single bellows fixed-fixed at both ends with rotational restraint vis-à-vis to the results presented in the reference [5].

Table 5.1 Comparison of Frequency Solutions for T=α and P $_{max}$

Mode #	**Exact [44] ω, rad/s**	**[28] ω, rad/s**	**Error, %**
1	3400.159	3400.334	0.005
2	6890.181	6890.356	0.002

The first two natural frequencies are calculated for the bellows data as given above using the frequency equation that is derived. It is seen that the error % of the frequency obtained from the exact method is about 0.005%. Therefore, as the percentage of error is well within the engineering accuracy it is found to be precise enough to estimate the natural frequency of single bellows expansion joint using this method.

Table 5.2: Natural Frequencies at P=0.0MPa for T=T_1=T_2 & N=1 to 4

T	N=1	N=2	N=3	N=4
0.01	1690.477	4845.901	8103.03	14431.45
0.1	1692.61	4847.431	8104.168	14432.18
1	1713.655	4862.628	8115.502	14439.46
10	1899.581	5005.034	8224.826	14511.13
10^2	2751.07	5851.544	12067.52	15109.97
10^3	3523.861	6909.163	13444.12	16590.34
10^4	3666.924	7123.951	13808.88	17018.88
10^5	3682.946	7154.077	13848.72	17066.6
10^6	3683.946	7156.511	13852.74	17071.41
10^7	3684.102	7156.754	13853.16	17071.89
10^8	3684.118	7156.778	13853.19	17071.94
10^9	3684.119	7156.781	13853.19	17071.95
10^{10}	3684.119	7156.781	13853.19	17071.95

Fig 5.2: Transverse Vibrations Single Bellows
(Lateral Frequency for P=0.0MPa & N=1 to 4 & T=T_1=T_2

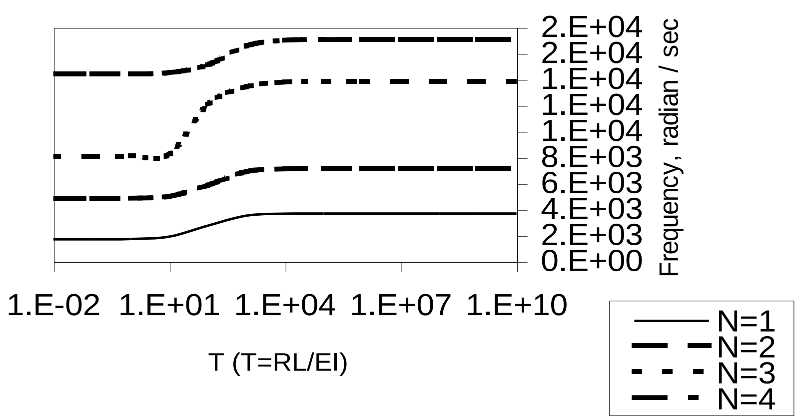
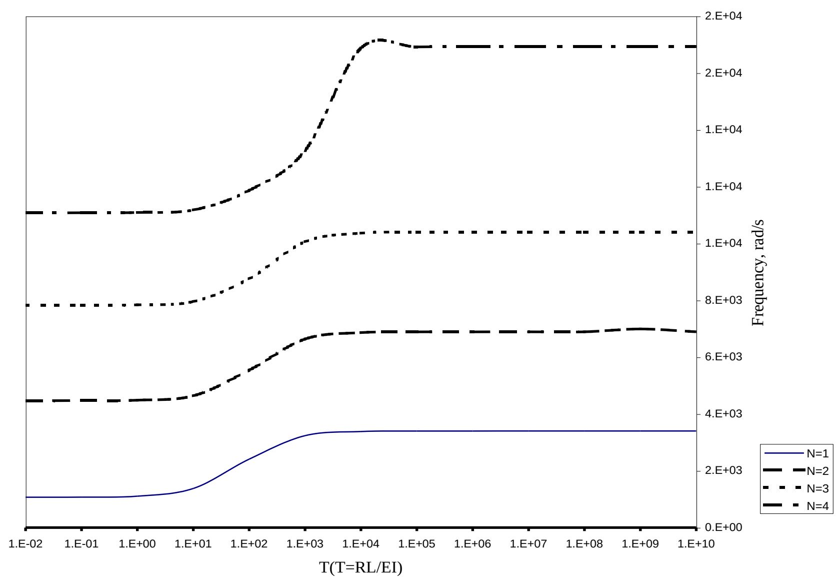


Table 5.3: Natural Frequencies at P=166.0MPa & T=T_1=T_2 & N=1 to 4

T	N=1	N=2	N=3	N=4
0.01	1069.425	4467.753	7828.299	11080.62
0.1	1072.798	4469.413	7829.476	11081.53
1	1105.697	4485.891	7841.207	11090.6
10	1375.915	4639.817	7954.354	11179.13
10^2	2408.615	5539.906	8754.995	11868.26
10^3	3235.598	6636.899	10063.38	13270.83
10^4	3382.595	6862.794	10364.6	16872.66
10^5	3398.391	6887.421	10397.61	16920.74
10^6	3399.982	6889.904	10400.94	16925.59
10^7	3400.142	6890.153	10401.27	16926.08
10^8	3400.142	6890.179	10401.3	16926.13
10^9	3400.16	6990.181	10401.31	16926.13
10^{10}	3400.16	6890.181	10401.31	16926.13

Fig 5.3 Transverse Vibrations Single Bellows
(Lateral Frequency for P=166.0MPa, T=T_1=T_2 and N=1 to 4)



The mode shapes are obtained for single bellows by varying the rotational restraint parameter T from 0.01 to 10 for both first and second modes of vibration (N=1&2) respectively. The ratio y/D is found out for ξ values between 0 to 0.5. The response for the first half of the mode is obtained and is found to be in symmetry for

the next half mode. The effect of rotational restraint T, on the response of single bellows with elastically restrained ends. Firstly, we observe the effect of variation of rotational restraint T on response of bellows. It is clearly seen that the response shows upward trend as ξ increases from 0 to 0.5 for the first half cycle of vibration. It is observed that at T=0.01, y/D is –175 and at T=10.0, y/D is –83 a rise by about 50%.

Table 5.4 Ratio y/D for different values of T and N=1

ξ	y/D, T=0.01	y/D T=0.1	y/D, T=1	y/D, T=10
0	0	0	0	0
0.1	-175	-44	-62	-83
0.2	-351	-87	-124	-165
0.3	-526	-133	-186	-248
0.4	-701	-177	-248	-330
0.5	-877	-221.5	-310	-412

Table 5.5 Ratio y/D for different values of T and N=2

ξ	y/D, T=0.01	y/D, T=0.1	y/D, T=1	y/D, T=10
0	0	0	0	0
0.1	4538	4462	4536	8330
0.2	9076.4	8925	9071	16659
0.3	13614.5	13387	13107	24989
0.4	18512.7	17849	18142	33318
0.5	22690.9	22312	22678	41647

Tables 5.4.and 5.5 presents values of y/D by varying the rotational restraint T and ξ=0 to 0.5. The same are plotted and shown in Fig. 5.4 and 5.5 respectively.

The response changes signs for the second mode of vibration. The response for T=10 and N2 is almost double when compared to the response at T=1.0 at ξ=0.5.

Fig 5.4 Mode Shape of Single Bellows with Elastically Restrained Ends for varying values of rotational restraint $T=T_1=T_2$ and P=0.0MPa, N=1

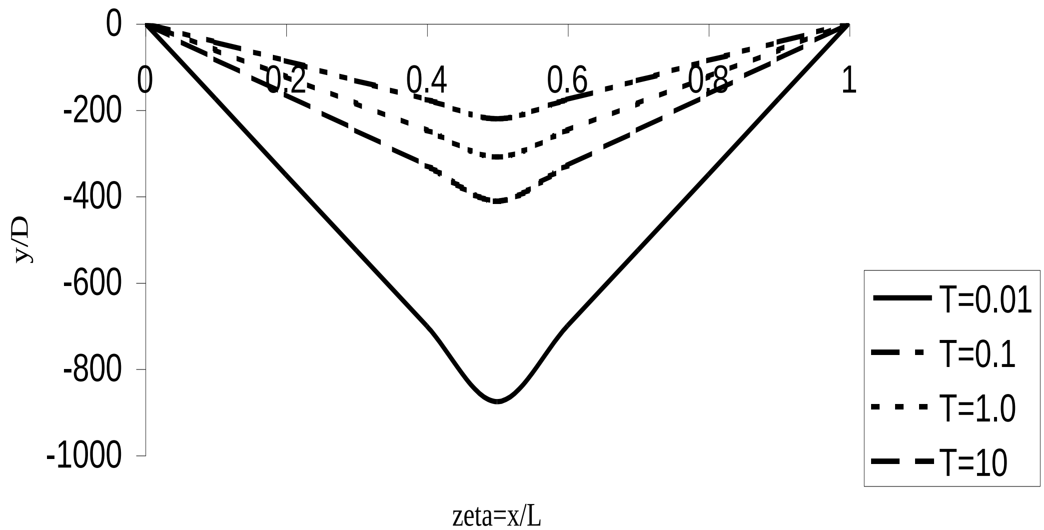
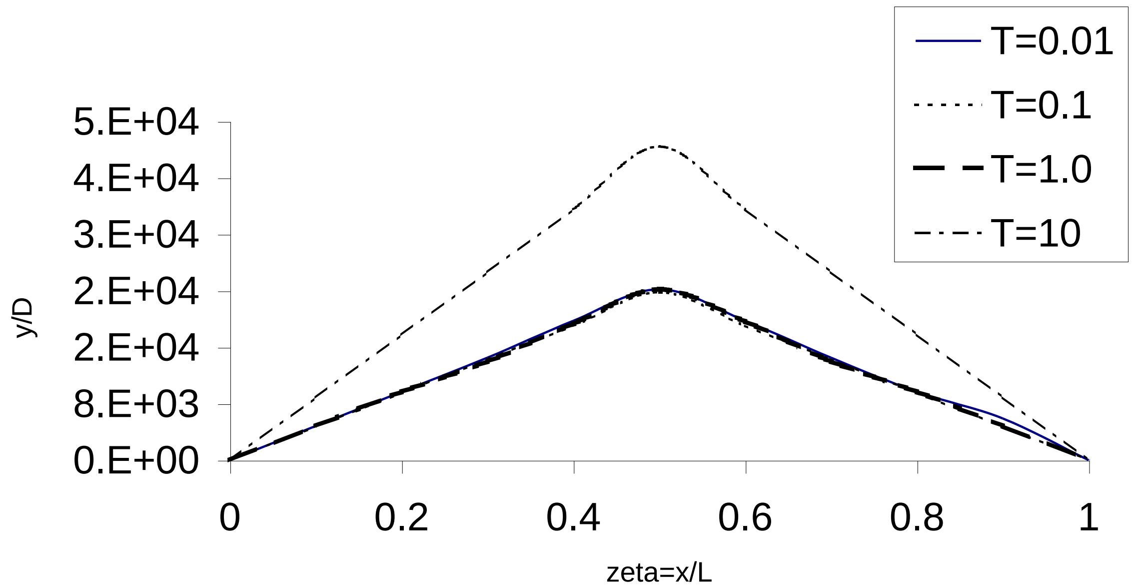


Fig 5.5 Mode Shape of Single Bellows with Elastically Restrained Ends for varying values of rotatioanl restraint $T=T_1=T_2$, P=0.0MPa & N=2



CHAPTER 6

FINITE ELEMENT ANALYSIS OF TRANSVERSE VIBRATIONS OF SINGLE BELLOWS EXPANSION JOINT RESTRAINED AGAINST ROTATION

6.1 Theoretical Background

This chapter presents the application of finite element method to the determination of the transverse frequencies of single type of bellows restrained against rotation on either end. The aim of the work here is to model the U shaped bellows using beam elements. The effect of rotatory inertia on the natural frequency is included in the beam element matrices.

Results obtained from the finite element method are then compared with the frequencies obtained by exact method for the first two modes of vibration. Experimental results are also used for comparison purpose. The effects of variation of the rotational restraint parameter, internal pressure and velocity on the natural frequencies are studied. The geometrical dimensions assumed in the present work are the same as considered by Jakubauskas.V.F.

6.2 Characteristics of Bellows

The various geometrical dimensional parameters of U-shaped bellows are given in Fig 1.2. Where r_1 is the meridional radius of the convolution root, r_2 is the meridional radius of the convolution crown and h is the convolution height. R_m is the

mean radius of the bellows, that is, the distance from the bellows centerline to mid convolution height, and t is the bellow material thickness. It is assumed that $t \ll r_1$, r_2 and $h \ll R_m$. With 'N' as number of convolutions the total length of the bellows is $L = 2(r_1+r_2)$ N. Therefore, with these assumptions bellows are considered as equivalent pipe / bar of radius R_m and wall thickness t.

6.3 Theory of Free Vibrations of Bellows

The matrix equation for the free vibration of bellows can be written as –

$$[M]\{q\} + [K]\{q\} = 0 \qquad (6.1)$$

Where

{q} -Generalized coordinates

[M] - Mass matrix

[K] - Elastic stiffness matrix

6.4 Finite Element Formulations of Elastic Stiffness Matrix

The strain energy U of a bellow element of length 'L' including the effect of rotatory inertia is given by the relation-

$$U = \tfrac{1}{2} EI \int_0^l (d^2w/dx^2)^2 .dx - \tfrac{1}{2} P\pi R_m^2 \int_0^l (dw/d\,x)^2 .dx \qquad (6.2)$$

Where

w - deflection of the bellow

EI -bending stiffness of bellow

P -internal pressure and

R_m -mean radius of bellows

Now on non-dimensionalizing and substituting the following relations, we get

$\eta = x / L$ and $\phi = y / L$ the above expression of strain energy becomes as follows –

$$U = \tfrac{1}{2}\, EI/L \int_0^l (d^2\phi/d\eta^2)^2 .d\eta - \tfrac{1}{2}\, P\pi R_m^2/ L \int_0^l (d\varphi/d\eta)^2 .d\eta \qquad (6.3)$$

Assuming a cubic polynomial expression for φ to be of the form –

$$\varphi = \sum a^r \eta_r \qquad (6.4)$$

Now substituting in to equation 6.1) and replacing the coefficient of a_r (r=0,1,2,3), the strain energy expression becomes -

$$U = \tfrac{1}{2}\, EI/l \{\xi\}^r [K] \{\xi\} \qquad (6.5)$$

Where ξ is the degree of freedom (DOF)

$$\therefore [K] = [k_e] - \Delta^2 [k_v] \qquad (6.6)$$

Where

$\Delta^2 = P\pi R_m^2 L^2 / EI$

[K] -Elastic stiffness matrix

$[k_e]$ -Velocity stiffness matrix and are defined as follows –

$$[k_e] = \frac{2EI}{L^3}\begin{bmatrix} 12 & -6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & -2L^2 & -6L & 4L^2 \end{bmatrix} \qquad (6.7)$$

$$[k_v] = \frac{\nabla^2}{30}\begin{bmatrix} -36 & -3L & -36 & -3L \\ 3L & 4L^2 & -3L & -L^2 \\ -36 & -36L & 36 & +3L \\ -3L & -3L^2 & +3L & 4L^2 \end{bmatrix} \qquad (6.8)$$

Where $\xi^T = \{\varphi_i, \varphi_i', \varphi_{i+1}, \varphi'_{i+1}\}$ (6.9)

6.5 Finite Element Formulation of Mass and Rotatory Inertia Matrices

The kinetic energy 'V' of a bellow element of length including the effects of rotatory inertia is given by –

$$V = \frac{1}{2} \rho A L^3 \int_0^l (\dot{\varphi})^2 d\eta + \frac{1}{2} \rho J L \int_0^l (\dot{\varphi}')^2 d\eta \tag{6.10}$$

The mass of bellows per unit length is computed by using the formula given in equation (6.11)-

$$M = \frac{\rho_b . 2\pi . R_m \{\pi(r_1 + r_2) + 2(h - r_1 - r_2)\} t}{2(r_1 + r_2)} \tag{6.11}$$

Where

ρ_b - mass density of bellows material

ρ_f - mass density of fluid flowing through bellows

A - area of cross-section of the bellow

Now substituting for φ from equation (6.2) and replacing the coefficient a_r (r=0,1,2,3) by the nodal coordinates the expression becomes-

$$T = \frac{1}{2} \rho A L^3 \{\dot{\xi}\}^T + [M]\{\dot{\xi}\} \tag{6.12}$$

Where

$$[M] = [M_1] + R \lambda^4 [M_2] \tag{6.13}$$

Where

$$R = \frac{J}{m_{tot}.L^2} \qquad \lambda^4 = \frac{m_{tot} L^4 \omega_n^2}{EI}$$

(6.14)

The mass moment of inertia of the bellow per unit length is given by-

$$J = Jxx = Jyy = \pi R_m^3 \left\{ \left(2\frac{h}{q} + 0.571 \right) t.\rho_b + \frac{h}{q}(2r_2 - t)\rho_f \right\}$$

(6.15)

Where R_m., h, q, t, r_1 and r_2 are already been defined earlier

Now the mass matrices M_1 and M_2 are defined as follows-

$$[M_1] = \frac{\rho AL}{420}\begin{bmatrix} 156 & -22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ -54 & -13L & 156 & -22L \\ 13L & -3L^2 & -22L & 4L^2 \end{bmatrix}$$

(6.15)

$$[M_2] = \frac{1}{420}\begin{bmatrix} -36 & -3L & -36 & -3L \\ -3L & 4L^2 & -3L & -L^2 \\ -36 & -3L & 36 & -3L \\ -3L & -L^2 & -3L & 4L^2 \end{bmatrix}$$

(6.16)

The matrix equation for free vibration of single bellows is given by-

$\{[K] - \lambda^4 [M]\} \{\xi\} = 0$ (6.17)

Where

$\lambda^4 = \{\rho A l^4 P_n^2\} / EI$

The matrix eigen value equation is solved for the bellows restrained against rotation using the following boundary conditions

6.6 Boundary Conditions

At the end 'A' - bellows are connected to a pipe nipple, and are considered to have a rotational stiffness of R_1 and R_2 at either end as shown in figure 5.1.

At the end A (x = 0)

$$X(x=0) = 0 \text{ and} \tag{6.18}$$

$$\frac{\partial^2 X(0)}{\partial x^2} = T_1 \frac{\partial X(0)}{\partial x} \tag{6.19}$$

At the end B (x = L) the boundary conditions are given by -

$$X(L) = 0 \text{ and} \tag{6.20}$$

$$\frac{\partial^2 X(L)}{\partial x^2} = T_2 \frac{\partial X(L)}{\partial x} \tag{6.21}$$

Where,

$$T_1 = R_1 L / EI \tag{6.22}$$

$$T_2 = R_2 L / EI \tag{6.23}$$

6.7 Results and Discussion

A single bellow with the following geometrical and physical parameters is considered –bellows length L = 0.0693m, mass moment of inertia per unit length, J = 0.001153kgm, EI = 5.078Nm and total mass m_{tot} = 5.138kg/m respectively. The

geometrical dimensions of bellows used are same as that considered by Jakubauskas.V.F (28).

The matrix eigenvalue equation is solved using the Jacobi's method for the bellows restrained against rotation and using the relevant boundary conditions. The stiffness matrix is notably modified to include the unequal restraints against rotation considered at either end.

The first two modes of transverse natural frequencies are obtained for maximum critical pressure of P=166. Firstly, it is seen that as internal pressure of the bellows increases, the lateral mode natural frequencies decrease. At P=0.0MPa and mode number N=1, natural frequency obtained is ω =1690.20rad/s and at P=166.0MPa and N=1, ω=1069.06 radian/s a drastic drop by about 37%. The same is true for even for other modes of vibration N=2 & 3 respectively.

It is also seen that as the value of rotational restraint T ($T=T_1=T_2$) -increases from 0.01 to 10^{10}, the frequencies tend to increase for all modes of vibration. As the rotational stiffness value $T\rightarrow\propto$, the frequency value increases by about 54.1% for N=1(P=0.0MPa) and by 68.56% for N=1 (P=166.0MPa) respectively. However, it is observed that there is no change in frequency and it becomes constant from $T=10^4$ onwards.

Table 6.1: Natural Frequencies for $T=T_1=T_2$ and N=1 to3 & P=0.0MPa

T	N=1	N=2	N=3
0.01	1690.2	4846.58	8112.22
0.1	1717.17	4866.0	8126.83
1.0	1927.65	5028.8	8252.68
10	2821.56	5938.55	9104.02
10^2	3544.04	6944.05	10343.6
10^3	3669.43	7138.60	10608.4
10^4	3682.84	7159.61	10637.1
10^5	3684.19	7161.73	10640.0
10^6	3684.33	7161.94	10640.3
10^7	3684.34	7161.94	10640.3
10^8	3684.34	7161.94	10640.3
10^9	3684.34	7161.94	10640.3
10^{10}	3684.34	7161.94	10640.3

Fig 6.1: Transverse Vibrations of Rotationally Restrained Single Bellows
T=0.01 to 10^{10}, P=0.0 MPa, RIX=0.04673 & V=0 m/s

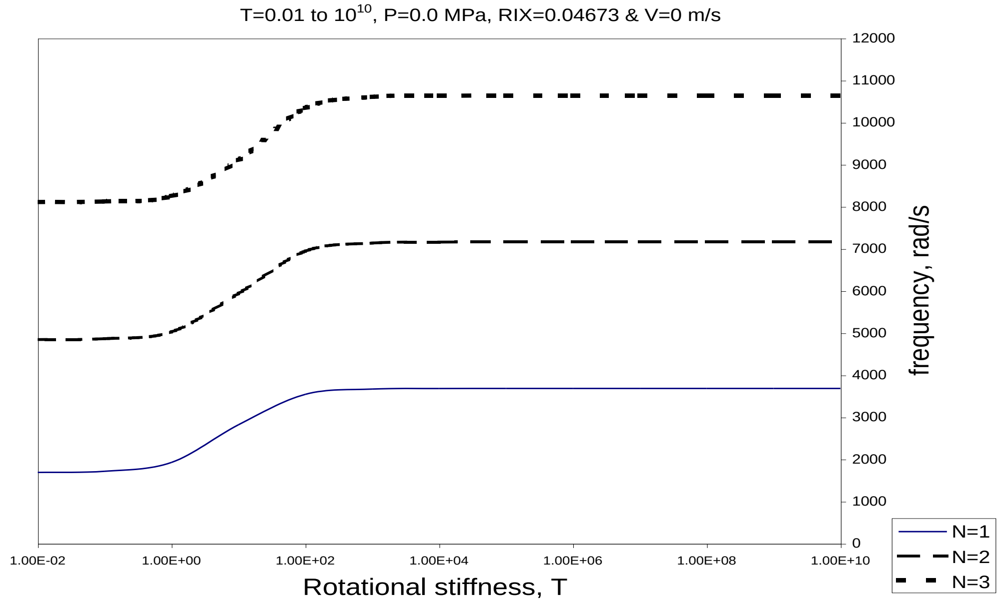


Table 6.2: Natural Frequencies for $T=T_1=T_2$ and N=1 to 3 & P=166.0MPa

T	N=1	N=2	N=3
0.01	1069.06	4468.55	7837.89
0.1	1111.2	4489.69	7852.98
1.0	1414.32	4665.5	7983.22
10	2486.92	5631.27	8862.16
10^2	3256.51	6672.85	10131.0
10^3	3385.25	6871.85	10400.1
10^4	3398.95	6893.31	10429.2
10^5	3400.33	6895.47	10432.1
10^6	3400.47	6895.69	10432.4
10^7	3400.48	6895.71	10432.5
10^8	3400.48	6895.71	10432.5
10^9	3400.48	6895.71	10432.5
10^{10}	3400.48	6895.71	10432.5

Fig 6.2: TransverseVibrations of Rotationally Restrained Singles Bellow for T=0.01 to 10^{10} , P =166.0 MPa & V=0 m/s, RIX=0.04673

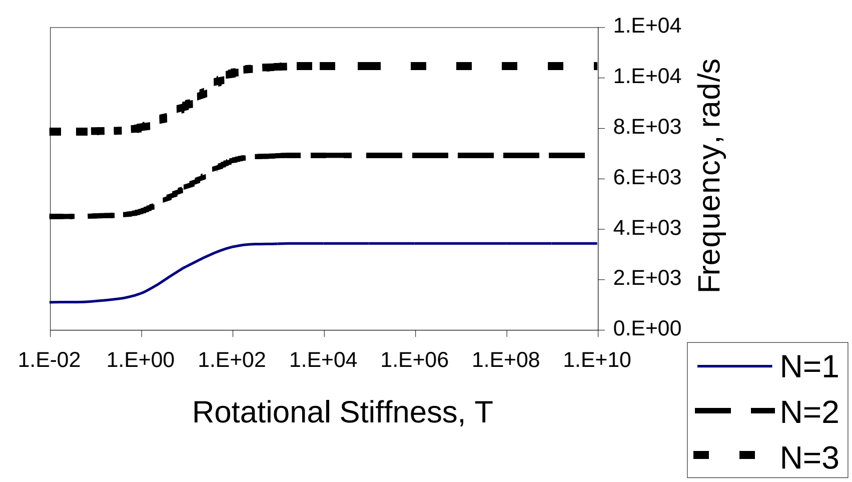


Table 6.3 gives a comparison of frequencies obtained by exact method with frequencies obtained using the finite element method for a single bellows elastically restrained at both ends.

Table 6.3 Comparison of Frequency for T=$\propto$ and P_{max}=166.0MPa

Mode #	[28] ω, rad/s	Exact ω, rad/s [44]	FEA [45] ω, rad/s
1	3400.334	3400.159	3400.48
2	6890.356	6890.181	6895.71

The first two natural frequencies are calculated for the bellows using the finite element method. It is seen that the results obtained by the finite element method are found to be in close agreement with the exact method. The percentage error in the frequency obtained is less is precise enough to estimate the natural frequency of single bellows expansion joint.

The effect of variation of velocity of flow at V= 1.0, 5.0 and 10 m/s on the first three frequencies is also studied. It is seen that for a constant value of rotational stiffness T = 10^{10} the frequency increases with increase in the velocity of flow and also with increase in mode numbers.

Table 6.4 Frequency at different velocities and constant T= 10^{10}

Mode Number	V=1.0m/s	V=5.0m/s	V=10.0m/s
	Frequency, Hz	Frequency, Hz	Frequency, Hz
N=1	0.08	2.0	7.4
N=2	0.16	4.0	15.5
N=3	0.25	6.0	23.7

It is seen that the effect of variation of pressure and velocity of flow on the natural frequency is marginal because the dimensions of bellows given in reference (4) are small. The effect of variation of pressure and flow velocity on the transverse natural frequencies for modes N=1, 2 & 3 is significant.

In order to demonstrate this clearly a large dimensioned bellow is considered that has the following dimensions -D_b= 1.2m, R_m = 0.6395m, L= 0.254m, EI=5449Nm^2, J=1485.36kgm and m_{tot}= 912.13kg/m respectively.

Firstly, the effect of velocity of flow for V=1.0 m/s and 5.0 m/s is studied on the natural frequency. It is found that for a constant value of $T \rightarrow \propto$ and velocity of flow increasing the natural frequency decreases. It is also observed that the frequency increases with increase in the modes of vibration N=1, 2 & 3. Table 6.5 shows the frequencies in radians/s obtained for different velocities of flow of liquid inside the bellow.

Table 6.5 Frequency in radian/s at T=∝ & Varying Velocities

Velocity	**V=1.0**	**V=2.0**	**V=3.0**	**V=4.0**	**V=5.0**
N=1	46.71	11.53	5.054	2.8	1.76
N=2	67.38	16.75	7.39	4.13	2.62
N=3	94.77	23.61	10.46	5.867	3.742

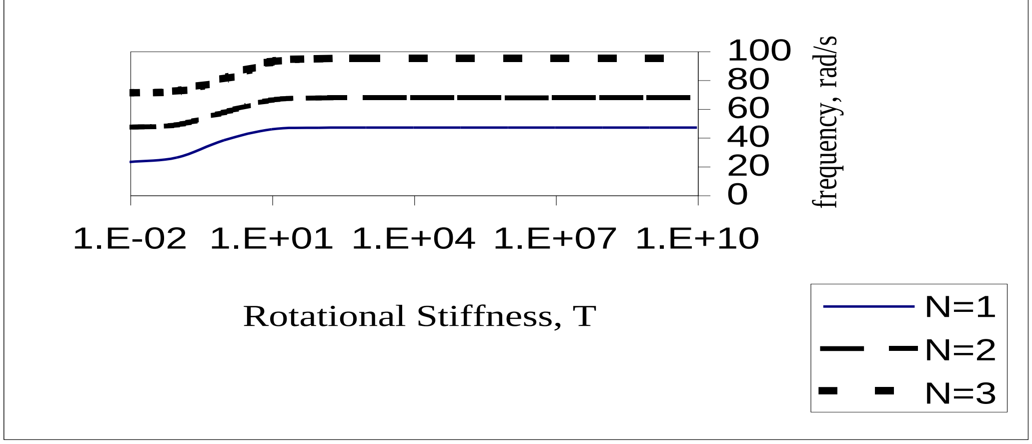
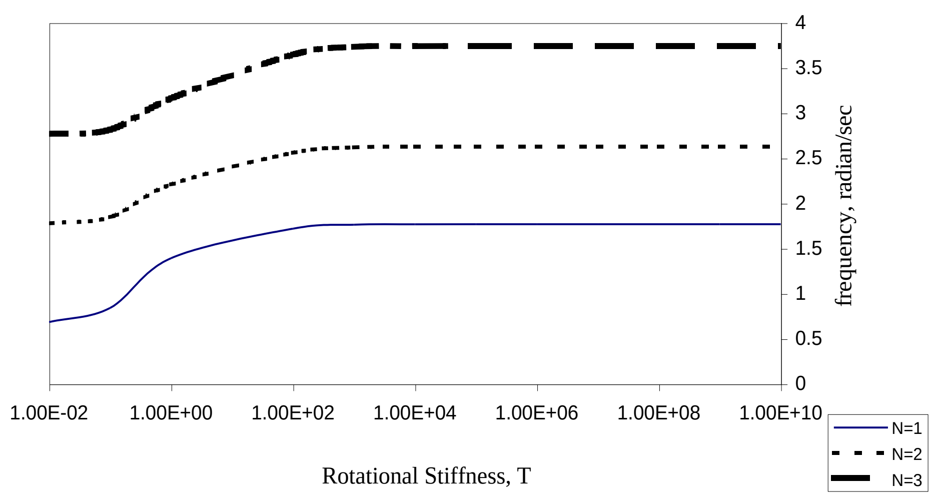


Fig 6.3: TransverseVibrations of Single Bellows Expansion Joint for V = 1.0m/s and N=1,2 &3 & T=infinity

Fig 6.4: Transverse Vibrations of Single Bellows Expansion joint for T=infinity and V=5.0m/s, N=1, 2 &3



Similarly, the effect of variation of internal pressure on natural frequency is investigated for the same geometrical dimensions of bellows. It is found that for $T \rightarrow \propto$ and pressure increasing from 10.0MPa to 150.0MPa, the transverse natural

frequencies for the first three modes of vibration decreases. Table 6.6 shows the frequencies obtained at different internal pressures of bellows.

It is seen that for P=10.0MPa & N=1, the percentage increase in frequency for T=0.01 to T=10^6 is 50% and then the frequency remains constant as T approaches infinity. At same pressure of 10.0MPa and N=2, the percentage increase in frequency for SR=0.01 to 10^6 is 30% and for N=3 the percentage increase in frequency is 25%. Similarly for a maximum pressure of 150.0MPa and N=1, the percentage increase in frequency for T=0.01 to 10^6 is 69%. At same pressure of 160.0MPa and N=2, the percentage increase in frequency for T=0.01 to 10^6 is 33% and for N=3 the percentage increase in frequency is 26%.

Table 6.6 Frequency in radian/s at T=∝ & Varying Pressures

Pressure	P=10MPa	P=50MPa	P=100MPa	P=150MPa
N=1	105.35	20.58	9.97	6.43
N=2	151.41	29.94	14.75	9.69
N=3	212.60	42.28	20.98	13.81

Fig 6.5: TransverseVibrations of Single Bellows Expansion Joint
for T=infinity and P=10.0 MPa, N=1,2&3

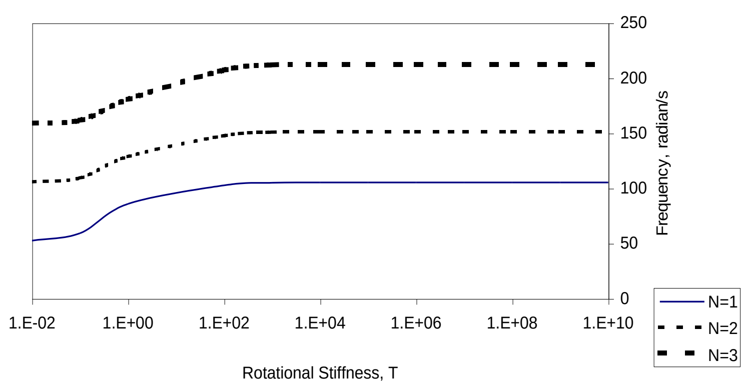


Fig 6.6: Transverse Vibrations of Single Bellows Expansion Joint
for T=infinity, P=50MPa and N=1,2 &3

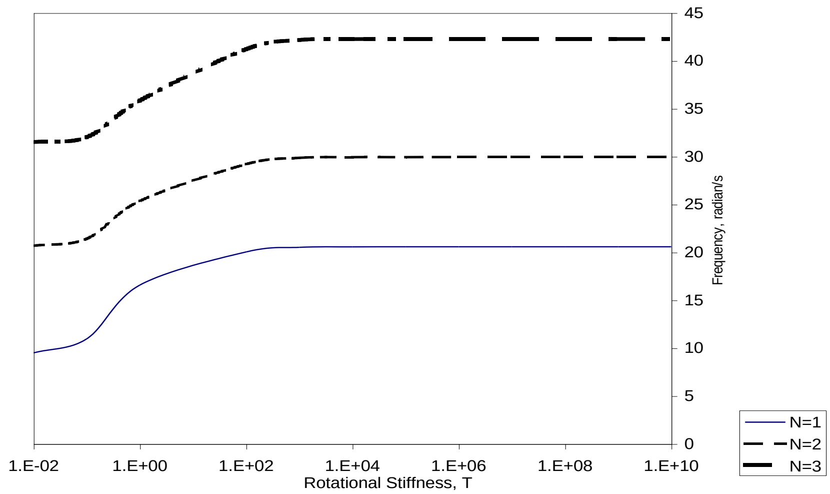


Fig 6.7: Transverse Vibrations of Single Bellows Expansion Joint
for T=infinity, P=100.0 MPa and N=1,2&3

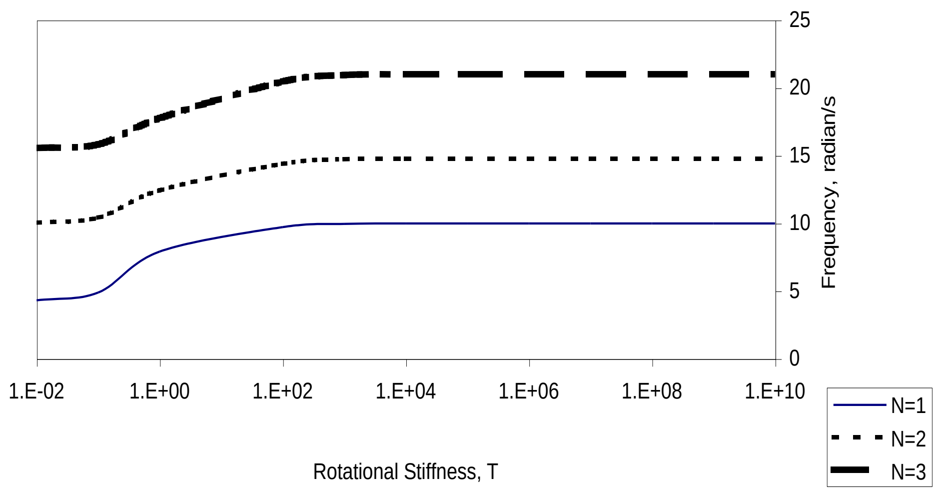
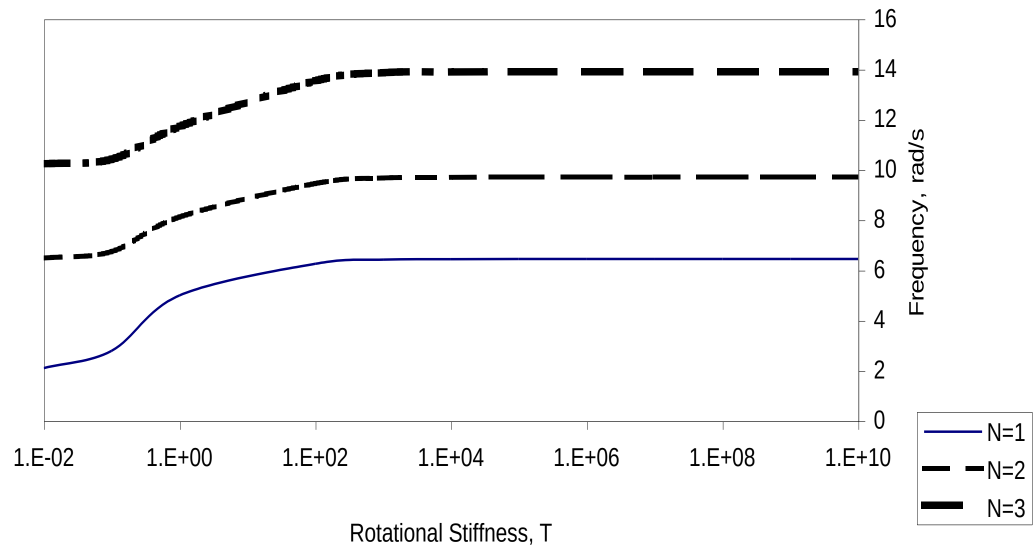


Fig 6.8: Transverse Vibrations of Single Bellows Expansion Joint for T=infinity and P=150.0 MPa and N=1,2 &3



Therefore, the finite element method developed gives fairly accurate results and is in close agreement to the exact and experimental results. The work leads to a logical conclusion that is as follows-

i. As the internal pressure of the bellow is doubled the transverse frequency of vibrations for the first three mode numbers is reduced by nearly 50%.

ii. It is seen that as the flow velocity is doubled the frequency decreases by about 4 times for all the first three modes of vibration.

CHAPTER 7

TRANSVERSE VIBRATIONS OF UNIVERSAL EXPANSION JOINT RESTRAINED AGAINST ROTATION (LATERAL MODE)

7.1 Universal Expansion Joint

A Universal double bellows type of expansion joint has two single bellows in series and separated between with a short pipe spool. Most studies of transverse vibration of these expansion joints consider the classical fixed-fixed end conditions for investigation of transverse mode shapes and natural frequency [46]. However, in real practice the bellow is subjected unequal rotations at the ends. Hence, the work aims at investigation of transverse vibrations of double bellows expansion joint by considering elastically restrained ends against rotation.

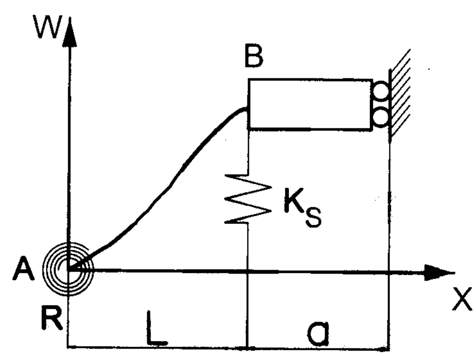


Fig 7.1 Mathematical Model of Double Bellows Expansion Joint

(Lateral Mode)

A theoretical model as shown in Figure 7.1 is developed based on the Timoshenko beam equation for lateral mode of vibration where L is the length of bellow and 'a' is the length of pipe. It neglects the effect of shear and includes the effect of rotatory inertia. An exact frequency equation is derived for the transverse vibrations of double bellows expansion joint that includes the variation of rotational restraint parameter, T.

The bisection method is used in computing the natural frequencies. They are then compared with bellow showing the effects of variation of elastic restraints and internal pressure on the first two modes of vibration.

7.2 The Differential Equation

The general form of the differential equation of vibration of bellows for single or double bellows expansion joint s given by

$$EI\frac{\partial^4 w}{\partial x^4} + P\pi R_m^2 \frac{\partial^2 w}{\partial x^2} - J\frac{\partial^4 w}{\partial x^2 \partial t^2} + m_{tot}\frac{\partial^2 w}{\partial t^2} = 0 \tag{7.1}$$

Where EI is the bending stiffness, P-internal pressure, J-mass moment of inertia per unit length, m_{tot} – total mass of bellows per unit length includes bellows material mass and fluid mass, x-axial coordinate, R_m- is the mean radius of bellow, w-deflection and t-time.

Using the technique of separation of variables, the lateral deflection of the bellows axis ‘w’ can be expressed as-

w (x, t) = X (x). T (t) (7.2)

T (t) = A e $^{i\omega t}$ (7.3)

Differentiating the above equations and substituting into differential equation (7.1) we get-

$$\frac{\partial^4 X}{\partial x^4} + \frac{\left(P\pi R_m{}^2 + J.w^2\right)}{EI}\frac{\partial^2 X}{\partial x^2} - w^2\frac{m_{tot}}{EI}X = 0$$

(7.4)

If $c = \sqrt{\frac{\left(P\pi R_m{}^2 + J\,\omega^2\right)}{2EI}}$ (7.5)

$$\lambda = \sqrt[4]{\frac{m_{tot}.\omega^2}{EI}}$$

(7.6)

$$\frac{d^4X}{dx^4} + 2c^2\frac{d^2X}{dx^2} - \lambda^4 X = 0$$

(7.7)

The general solution of the equation is given by-

X(x) = A sinh αx + B cosh αx + C sinβx+ D cosβx (7.8)

The first three derivatives of equation (7.8) are given as follows-

$\frac{d(x)}{dx}$ = Aα cosh αx + $\beta\alpha$ sinh αx + Cβ cos β + Dβ sinβx (7.9)

$$\frac{d^2(x)}{dx^2} = A\alpha^2 \sinh \alpha x + B\alpha^2 \cosh \alpha x - C\beta^2 \sin\beta x - D\beta^2 \cos\beta x \tag{7.10}$$

$$\frac{d^3(x)}{dx^3} = A\alpha^3 \cosh \alpha x + B\alpha^3 \sinh \alpha x - C\beta^3 \cos\beta x + D\beta^3 \sin\beta x \tag{7.11}$$

$$\alpha = \sqrt{-c^2 + \sqrt{c^4 + \lambda^4}}$$

Where the roots of the equation are α & β and their values are given by –

(7.12)

$$\beta = \sqrt{c^2 + \sqrt{c^4 + \lambda^4}}$$

(7.13)

Where A, B, C, D are arbitrary constants

7.3 Derivation of Frequency Equation for Transverse Vibrations of Elastically Restrained Double Bellows Expansion Joint in Lateral Mode

A case of vibration of bellows in lateral mode is shown in figure 7.1. It is seen that the pipe has pure transnational motion due to the geometry and physical symmetry of the system provided. The coriolis component of force acting on the bellows from the fluid flowing inside is neglected.

As mathematical approximation, one half of the system is considered with its left end having a rotational stiffness 'R' and right end fixed to the vertical rollers.

The boundary conditions for such a system is given by-

$$EI\frac{\partial^2 X(0)}{\partial x^2} = R\frac{\partial X(0)}{\partial x}$$

Deflection is zero at w (0, t) = 0 at x=0 (7.14)

(7.15)

(7.16)

$$\therefore \frac{\partial^2 X(0)}{\partial x^2} = \frac{R}{EI}\frac{\partial X(0)}{\partial x}$$

If T=RL/EI, we get

$$\frac{\partial^2 X(0)}{\partial x^2} = T.\frac{\partial X(0)}{\partial x}$$

(7.17)

As the end B does not rotate the third boundary condition, slope is zero and given by

$$\frac{\partial w\,(L,t)}{\partial x} = 0 \tag{7.18}$$

The fourth boundary condition at end 'B' is the shear force Q (L, t) of the bellows and is given by –

$$\frac{\partial^3 W\,(L,t)}{\partial x^3} = \frac{\left\{M_s + \left(m_p + m_{f3}\right)a\right\}}{EI}\frac{\partial^2 W\,(L,t)}{\partial t^2} + \frac{k_s W(L,t)}{EI} \tag{7.19}$$

$$\frac{d^3W(L)}{dx^3}=\frac{-W^2\left\{M_s+\left(m_p+m_{f3}\right)a\right\}}{EI}*X(L)+\frac{k_s}{EI}(L) \tag{7.20}$$

If $$b=\frac{-W^2\left\{M_s+\left(m_p+m_{f3}\right)a\right\}}{EI}-\frac{k_s\,X(L)}{EI} \tag{7.21}$$

M_s is the equivalent lateral support mass, k_s is the equivalent spring stiffness of the lateral support, m_p is the mass per unit length of the connecting pipe of length ‘a’ and m_{f3} is the mass of the fluid per unit length in the connecting pipe.

Substitution of the general solution (7.8) and its derivatives into boundary condition expressions (7.14), (7.15), (7.16) and (7.17) will give a set of linear equations with respect to the constants A, B, C and D.

$$B + D = 0 \tag{7.22}$$

$$\left\{B(\alpha L)^2 - D(\beta L)^2 - T(A\,\alpha L + C\,\beta L)\right\} = 0 \tag{7.23}$$

$$A\alpha \cosh \alpha L + B\alpha \sinh \alpha L + C\beta \cos\beta L - D\beta \sin\beta L = 0 \tag{7.24}$$

$$[\alpha^3 \cosh \alpha L + b \sinh \alpha L]\, A + [\alpha^3 \sinh\alpha L + b \cosh \alpha L]\, B - [\beta^3 \cos\beta L - b \sin \beta L]\, C + [\beta^3 \sin\beta L + b \cos \beta L]\, D = 0 \tag{7.25}$$

Substituting c_1, c_2, c_3, c_4 for the terms in the brackets in equation (7.23) we get –

$$c_1 = \alpha^3 \cosh \alpha L + b \sinh \alpha L \tag{7.26}$$

$$c_2 = \alpha^3 \sinh \alpha L + b \cosh \alpha L \tag{7.27}$$

$$c_3 = \beta^3 \cos\beta L - b \sin\beta L \tag{7.28}$$

$c_4 = \beta^3 \sin\beta L + b \cos \beta L$ (7.29)

For a non-trival solution the determinant formed by the coefficients of the system of algebraic equations above must be equal to zero –

The expansion of the above determinant results in the frequency equation of double bellows in lateral mode-

$$\begin{vmatrix} 0 & 1 & 0 & 1 \\ +T(\alpha) & (\alpha)^2 & +T(\beta) & +(\beta)^2 \\ \alpha\cosh(\alpha L) & \alpha\sinh(\alpha L) & \beta\cos\beta L & -\beta\sin\beta L \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix} \begin{vmatrix} A \\ B \\ C \\ D \end{vmatrix} = 0 \tag{7.30}$$

The frequency equation is

$$T\ \{b\ [(\alpha^2 - \beta^2) \sin\beta L \sinh\alpha L + 2\alpha\beta\ (1- \cos\beta L \cosh\alpha L)\} - \alpha\beta\ (\alpha^2 + \beta^2)\ (\alpha \cos\beta L \sinh\alpha L + \beta \sin\beta L \cosh\alpha L)\} + b\ (\alpha^2 + \beta^2)\ (\alpha \sin\beta L \cosh\alpha L - \beta \cos\beta L \sinh\alpha L)\} - \alpha\beta(\alpha^2 + \beta^2) \cos\beta L \cosh\beta L = 0 \tag{7.31}$$

Table 7.1: Exact Natural Frequency for T=∞, N=1,2 and M_s=0

Spring Support Stiffness k_s	Frequency N=1, rad/s	Frequency N=2, rad/s
10^2	331.4023	3500.686
10^3	333.41187	3500.698
10^4	352.8724	3500.812
10^5	507.8993	3501.972
10^6	1256.9545	3515.098
10^7	3204.5183	4195.031
10^8	3392.1072	6856.604
10^9	3399.4055	13673.99
10^{10}	3400.1597	13678.91

Table 7.2: Exact Natural Frequency for N=1, 2 and T=∝ & k_s=0

Support mass M_s	Frequency, rad/s N=1	Frequency, rad/s N=2
1	210.103	3439.571
10	83.1873	3406.248
20	59.7721	3403.299
30	49.0707	3402.274
40	42.6135	3401.754
50	38.8901	3401.144
60	34.8901	3401.129
70	32.3276	3401.074
80	30.2577	3400.963
90	28.5405	3400.875
100	27.0859	3400.804

Fig 7.2 Exact Transverse Vibrations of Double Bellows
For N=2 and k_s=0

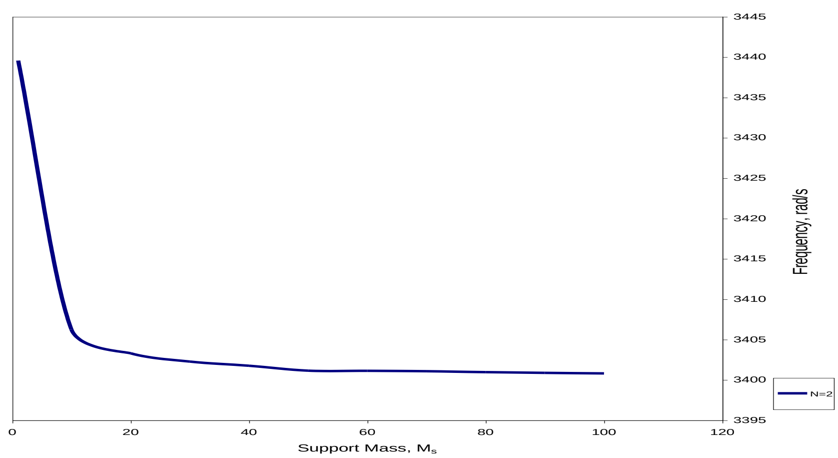
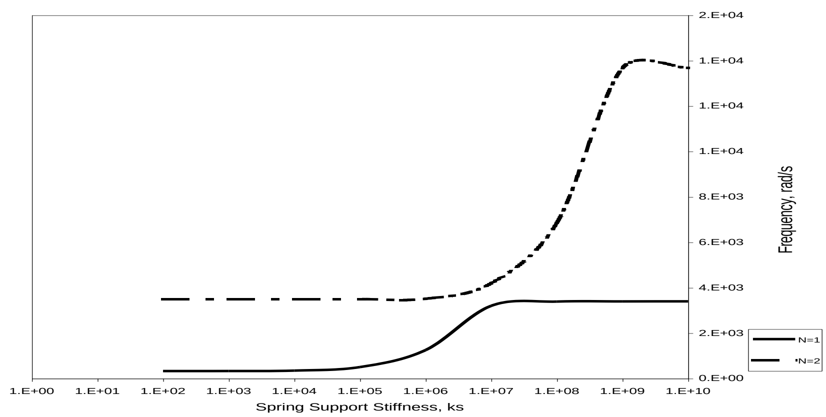


Fig 7.3 Exact Transverse Vibrations (Lateral) of Double Bellows For N=1 & 2 and M_s=0



7.4 Mode Shape Expression

The boundary conditions of double bellows subjected to lateral mode of vibration are as follows-

$$\text{At} X(x) = 0,$$

$$\frac{\partial^2 X}{\partial x^2} = \frac{RL}{EI}\frac{\partial X}{\partial x}$$

$$\frac{\partial^2 X}{\partial x^2} = T_1 \frac{\partial X}{\partial x}$$

where

$$\frac{RL}{EI} = T_1$$

$$\frac{dx(L)}{dx} = 0 \text{ at } x = L$$

$$A\alpha \cosh \alpha L + B\alpha \sinh \alpha L + C\beta \cos \beta L - D\beta \sin \beta L = 0$$

(7.32)

And Substituting, in equation (7.32) we get-

B+D=0 $\therefore$ B= –D

(7.33)

$$L^2(B\alpha^2 - D\beta^2) = \frac{R_1 L}{EI}(\alpha LA + \beta LC)$$

$$B(\alpha L)^2 - D(\beta L)^2 - T_1(A\alpha L + C\beta L) = 0$$

Substituting in equation (7.33), we get-

$$\frac{d^3(X)}{dx^3} = 0$$

i.e.

$$\alpha^3(\cosh \alpha L + b \sinh \alpha L)A + (\alpha^3 \sinh \alpha L + b \cosh \alpha L)B - (\beta^3 \cos \beta L - b \sin \beta L)C$$

$$+ (\beta^3 \sin \beta L + b \cos \beta L)D = 0, \text{where}$$

$$c_1 = \alpha^3 \cosh \alpha L + b \sinh \alpha L$$

$$c_2 = \alpha^3 \sinh \alpha L + b \cosh \alpha L$$

$$c_3 = \beta^3 \cos \beta L + b \sin \beta L$$

$$c_4 = \beta^3 \sin \beta L + b \cos \beta L$$

$$-D(\alpha L)^2 - D(\beta L)^2 - T_1(A\alpha L + C\beta L) = 0$$

$$-D\left[(\alpha L)^2 + (\beta L)^2\right] - T_1(A\alpha L + C\beta L) = 0$$

$$-T_1(A\alpha L + C\beta L) = D\left[(\alpha L)^2 + (\beta L)^2\right] \quad (7.34)$$

Now,

$$A\alpha\cosh\alpha L + B\alpha\sinh\,\alpha L + C\beta\cos\beta L - D\beta\sin\,\beta L = 0$$

By substituting B= -D the above equation can be written as-

$$A\alpha\cosh\alpha L - D\alpha\sinh\,\alpha L + C\beta\cos\beta L - D\beta\sin\,\beta L = 0$$

$$A\alpha\cosh\alpha L + C\beta\cos\beta L - D[\alpha\sinh\,\alpha L + \beta\sin\,\beta L] = 0$$

$$A\alpha\cosh\alpha L + C\beta\cos\beta L = D[\alpha\sinh\,\alpha L + \beta\sin\,\beta L] \qquad (7.36)$$

Multiplying equation (7.34) by cos hαL and equation (7.36) by T_1L and canceling the like terms A and C, we get the ratios C/D and A/D respectively-

$$\cosh\alpha L \ \text{x} \ -T_1(A\alpha L + C\beta L) = D\left|(\alpha L)^2 + (\beta L)^2\right|$$

$$T_1 L \qquad \text{x} \ A\alpha\cosh\alpha L + C\beta\cos\beta L = D(\alpha\sinh\,\alpha L + \beta\sin\,\beta L)$$

$$-T_1\ A\alpha L\cosh\alpha L + C\beta L\cosh\alpha L = D\left|(\alpha L)^2 + (\beta L)^2\right|\cosh\alpha L$$

$$-T_1\ A\alpha L\cosh\alpha L + C\beta L\cosh\alpha L = D[\alpha\sinh\,\alpha L + \beta\sinh\,\beta L]T_1 L$$

$$C\beta L[\cosh\alpha L + T_1\cos\beta L] = D\left\{\left|(\alpha L)^2\cosh\alpha L + (\beta L)^2\cosh\alpha L\right| + [\alpha L\sinh\,\alpha L + \beta L\sinh\,\beta L]T_1\right\} \qquad (7.37)$$

$$C' = \frac{C}{D} = \left\{\frac{\left[(\alpha L)^2\cosh\alpha L + (\beta L)^2\cosh\alpha L\right] - [\alpha\sinh\,\alpha L + \beta\sinh\,\beta L]T_1}{\beta L(\cosh\alpha L + T_1\cos\beta L)}\right\}$$

Now finding out the ratio A/D-

$$\cos\beta L \ \text{x} \ T_1(A\alpha L + C\beta L) = D\left|(\alpha L)^2 + (\beta L)^2\right|$$

$$L \ \text{x} \ A\alpha\cosh\alpha L + C\beta\cos\beta L = D[\alpha\sinh\,\alpha L + \beta\sin\,\beta L]$$

$$-T_1\ \alpha LA\cos\beta L + C\beta L\cos\beta L = D\left|(\alpha L)^2 + (\beta L)^2\right|\cos\beta L$$

$$A\alpha L\cosh\alpha L + C\beta L\cos\beta L = D[\alpha\sinh\,\alpha L + \beta\sinh\,\beta L]L$$

$$-\left[T_1\alpha L\cos\beta L+\alpha L\cosh\alpha L\right]A=D\left\{\left[(\alpha L)^2\cos\beta L+(\beta L)^2\cos\beta L\right]-\left[\alpha L\sinh\ \alpha L+\beta L\sin\ \beta L\right]\right\}$$
(7.38)

$$A'=\frac{A}{D}=\left\{\frac{\left[(\alpha L)^2\cos\beta L+(\beta L)^2\cos\beta L\right]-\left[\alpha\sinh\ \alpha L+\beta\sinh\ \beta L\right]}{\left(T_1\alpha L\cos\beta L+\alpha L\cosh\alpha L\right)}\right\}$$

$$\frac{B}{D}=-1$$

Substituting

$$y_{max}=D\left[\frac{A}{D}\alpha\cosh\alpha L\xi+\frac{B}{D}\alpha\sinh\ \alpha L\xi+\frac{C}{D}\beta\cos\beta L\xi-\beta\sin\beta L\xi\right]=0$$

$$\xi=\frac{C}{L}$$

$$\frac{y_{max}}{D}=A'\alpha\cosh\alpha L(\xi)-\alpha\sinh\ \alpha L\xi+C'\beta\cos\beta L\xi-\beta\sin\ \beta L\xi$$

Where

$$A'=\frac{A}{D}=\left\{\frac{\left[(\alpha L)^2\cos\beta L+(\beta L)^2\cos\beta L\right]-\left[\alpha L\sinh\ \alpha L+\beta L\sin\ \beta L\right]}{\left(T_1\alpha L\cos\beta L+\alpha L\cosh\alpha L\right)}\right\}$$

$$C'=\frac{C}{D}=\left\{\frac{\left[(\alpha L)^2\cosh\alpha L+(\beta L)^2\cosh\alpha L\right]+T_1\left[\alpha L\sinh\ \alpha L+\beta L\sin\ \beta L\right]}{\beta L\left(\cosh\alpha L+T_1\cos\beta L\right)}\right\}$$

Therefore, the final mode shape equation can be written as –

$$\frac{y}{D}=A'\alpha\cosh\alpha L\xi-\alpha\sinh\ \alpha L\xi+C'\beta\cos\beta L\xi-\beta\sin\ \beta L\xi$$
(7.39)

Where A′ and C′ have already been defined above.

7.5 Results and Discussion

A double bellows expansion joint having the following geometrical dimensions is taken and is similar to the bellows considered by Jakubauskas.V.F. Bellow length L = 0.0693m, mass moment of inertia per unit length J =

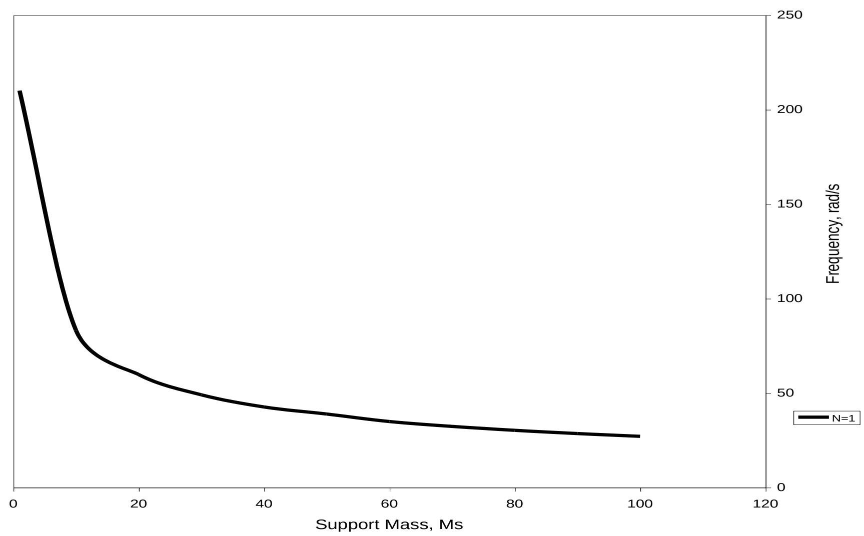
0.001153kgm, EI = 5.078 Nm^2, total bellows mass m_{tot} = 5.13kg/m, total connecting pipe mass, m_p+m_{f3} = 5.0kg/m and assuming a = L.

According to [1], the maximum allowable pressure in bellow is –

$$P_{max} = \frac{\pi k P}{6.666 L^2} \tag{7.32}$$

Where k is the equivalent axial stiffness of bellows –

Fig 7.4 Exact Transverse Vibrations of Double Bellows
For N=1, ks=0



Substitution of the above numerical values into the expressions 7.4 & 7.5, we get –

$$c = \sqrt{616.49 + 0.0001135\omega^2} \tag{7.33}$$

$$\lambda = 1.0029\sqrt{\omega} \tag{7.34}$$

$$b = 0.10235\omega^2 \tag{7.35}$$

The study involves the effect of varying the equivalent support stiffness and mass on the frequency of vibration. Using expressions in (7.12), (7.13), (7.32), (7.33), (7.34) and (7.35) the frequency equation (7.31) is solved by applying bisection method. The results are presented in Tables 7.1 and 7.2 respectively.

Table 7.1 presents the frequencies obtained by varying the support stiffness k_s for modes of vibration N=1 and 2. It is seen that as stiffness increases from a value of 10^2 to 10^{15}, frequency increases from 331.40rad/s to 3400.1597rad/s, a steep rise by almost 90%. The frequency attains almost a constant value from 10^8 onwards.

The effect of varying the support mass on the frequency of vibration is also studied and the results presented in Table 7.2. It is seen that there is a steep drop of 87% in frequency as the support mass increases from 1.0 to 100 N=1 and by a small drop of 1% for N=2.

Table 7.3 Exact Lateral Mode Frequency for T=$\propto$ and M_s =0

S. No	**k_s, spring stiffness**	**Frequency, ω (rad/s) N=1**	**Frequency, ω (rad/s) N=2**
1	10^2	331.4	3500.68
2	10^{10}	3400.1597	13678.906

Table 7.4 Exact Lateral Mode Frequency for T=$\propto$ and k_s =0

S. No	**M_s, support mass**	**Frequency, ω (rad/s) N=1**	**Frequency, ω (rad/s) N=2**
1	1	210.0	3439.5
2	100	27.0	3400.8

Fig 7.5 First Mode Shape of Double Bellows with Elastically Restrained Ends for Rotational RestraintT=infinity, Support stiffness ks at 10^2& infinity, Support mass, M_s=0 and N=1

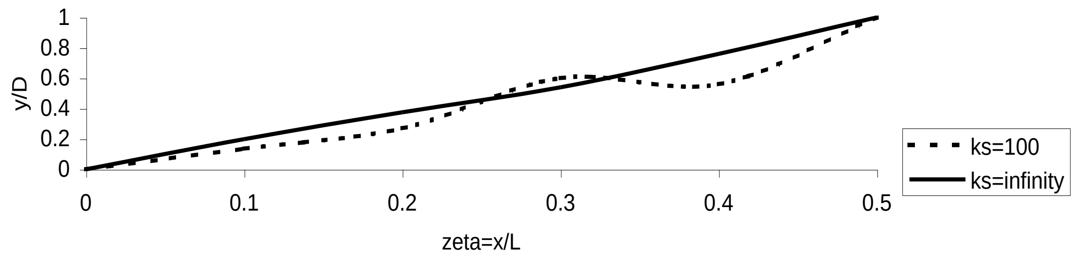


Fig 7.6 Second Mode Shape of Double Bellows with Elastically Restrained Ends at T=infinity, Support stiffness ks at 10^2 & infinity, Support mass Ms=0 and N=2

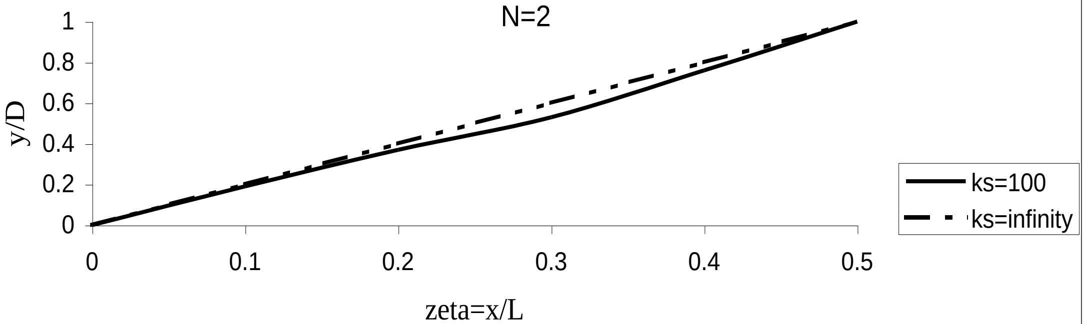


Table 7.5: y/D at T=$\propto$, M_s=0 and N=1

ξ	**y/D, k_s=100**	**y/D, k_s=$\propto$**
0.1	0.9×10^{-8}	-0.14×10^{-6}
0.2	0.18×10^{-7}	-0.27×10^{-6}
0.3	0.4×10^{-7}	-0.39×10^{-6}
0.4	0.37×10^{-7}	-0.55×10^{-6}
0.5	0.66×10^{-7}	-0.72×10^{-6}

Table 7.6: y/D at T=$\propto$, M_s=0 and N=2

ξ	**y/D, k_s=100**	**y/D, k_s=$\propto$**
0.1	0.83×10^{-8}	0.17×10^{-5}
0.2	0.16×10^{-7}	0.34×10^{-5}
0.3	0.23×10^{-7}	0.51×10^{-5}
0.4	0.33×10^{-7}	0.68×10^{-5}
0.5	0.43×10^{-7}	0.85×10^{-5}

Figures 7.5 and 7.6 indicate the mode shapes of double bellows with elastically restrained ends. The support stiffness k_s is varied and its effect on the mode shape is studied at ks=100 and $k_s=\propto$, and by keeping the support mass, $M_s=0$ and $T=\propto$. The values y/D are obtained for two modes of vibration, N=1 & 2 respectively. The ordinate in Fig 7.5 & 7.6 is the ratio of minimum to maximum y/D plotted for different values of ξ ranging from 0 to 0.5 for the first half mode. It is found that as ξ varies between 0 to 1 and support stiffness k_s increases, the mode shape follows an up trend with a fall and rise at ξ=0.2 and 0.3 and remains a straight line approaching infinity. This phenomenon is observed for both the modes of vibration N=1 and 2 respectively.

Fig 7.7 Mode Shape of Double Bellows with Elastically Restrained Ends for $T=T_1=T_2$ and varying from 0.01 to 100, Support stiffness k_s=100 and Support mass M_s=0

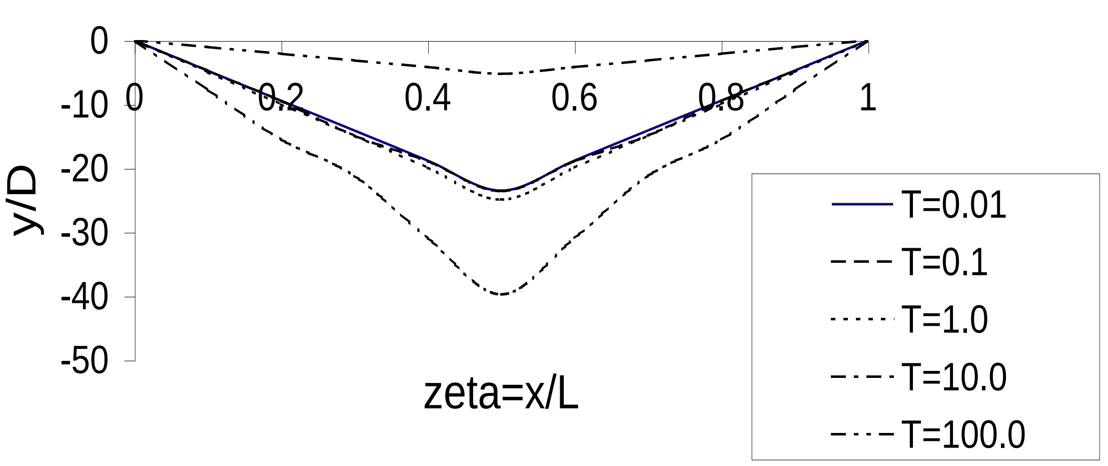


Figure 7.7, shows the mode shape of double bellows by varying the rotational restraint parameter T, from 0.01 to 100. The support stiffness k_s is kept constant at 100 and support mass, M_s =0. The plot between ξ and ratio y/D- indicate that as T increases, the ratio y/D decreases. It is observed that there is a sharp rise in y/D at T=100, and ξ=0.1. However, no significant change is observed in the ratio y/D by varying T from 0.01 to 10.

Table 7.7: y/D for T=0.01 to 100

ξ	y/D,T=0.01	y/D,T=0.1	y/D,T=1.0	y/D,T=10	y/D,T=100
0	0	0	0	0	0
0.1	-4.73	-4.73	-4.98	-7.74	-1.04
0.2	-9.42	-9.47	-9.97	-15.5	-2.08
0.3	-14.1	-14.9	-14.9	-21.2	-3.13
0.4	-18.8	-18.9	-19.9	-30.9	-4.17
0.5	-23.5	-23.6	-24.9	-39.74	-5.21
0.6	-18.8	-18.9	-19.9	-30.9	-4.17
0.7	-14.1	-14.9	-14.9	-21.2	-3.13
0.8	-9.42	-9.47	-9.97	-15.5	-2.08
0.9	-4.73	-4.73	-4.98	-7.74	-1.04
1	0	0	0	0	0

CHAPTER 8

TRANSVERSE VIBRATIONS OF UNIVERSAL EXPANSION JOINT RESTRAINED AGAINST ROTATION (ROCKING MODE)

8.1 Derivation of Frequency Equation for Transverse Vibrations of Elastically Restrained Double Bellows Expansion Joint in Rocking Mode

In case of the vibration of bellows in rocking mode the middle point of the connecting pipe does not translate because of the geometrical and physical symmetry of the system with respect to the imaginary vertical axis. The coriolis forces acting on the bellows from the fluid inside are neglected. Therefore, as a mathematical approximation, one half of the physical system can be taken with its left end fixed and right end simply supported in the middle of the connecting pipe as shown in fig 8.3

Therefore, one half of the system can be taken with its left end 'A' fixed and having a rotational stiffness 'R' while the right end 'B' is simply supported and held at the middle of the connecting pipe.

8.2 Boundary Conditions for Transverse Vibrations of DBEJ Rocking Mode

As shown from figure 8.1 the boundary conditions at the end 'A & B' are given by –

$$\frac{dX(L)}{dx} = -\frac{X(L)}{a}$$

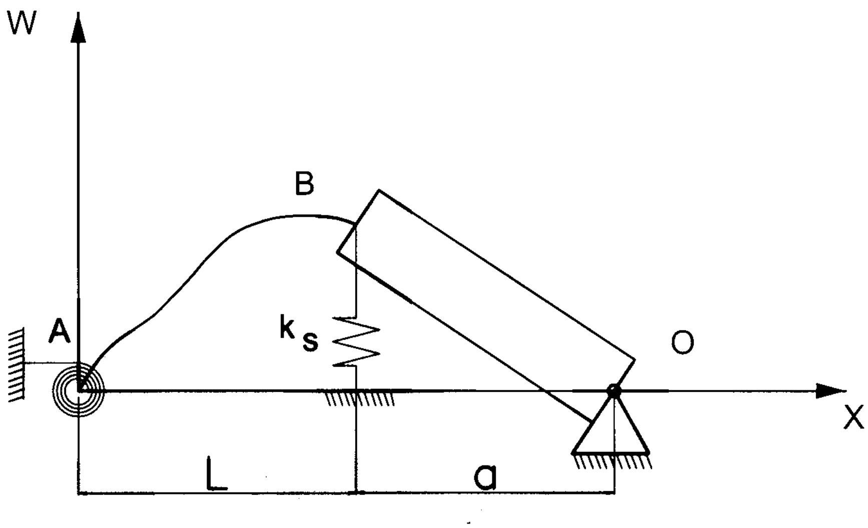


Fig 8.1 Mathematical Model of Double Bellows Expansion Joint (Rocking Mode)

$$\frac{d^3X(L)}{dx^3} = -\frac{d^2X(L)}{dx^3}\frac{L}{a} - \frac{P\pi R_m^2}{EI}\frac{dX(L)}{dx} - \frac{J\omega^2}{EI}\frac{dX(L)}{dx} - \omega^2\frac{Jtot}{EI a^2}X(L) + \frac{ks}{EI}X(L)$$

w (0, t) = 0 at x = 0 (8.1)

B + D = 0 and (8.2)

EI ($\beta\alpha^2$ - $D\beta^2$) = R ($A\alpha$ - $C\beta$) (8.3)

$$\frac{\partial w(L,t)}{\partial x} = \frac{1}{a} w(L,t) \qquad (8.4)$$

a[$A\alpha$ cosh αL + $B\alpha$ sinh αL + $C\beta$cos βL - $D\beta$sinβL] + [A sinh αL + B cosh αL + C sinβL + D cosβL] = 0 (8.5)

(α a cosh αL + sinh αL) A + (αa sinhαL + cosh αL) B + (βa cosβL + sinβL) C – (βa sinβL - cosβL) D = 0 (8.6)

Where

d_1 = (αa cosh αL + sinh αL) (8.7)

d_2 = (αa sinhαL + cosh αL) (8.8)

d_3 = (βa cosβL + sinβL) (8.9)

d_4 = (βa sinβL - cosβL) (8.10)

Another boundary condition at end 'B' was derived from the rotation differential equation for one-half of connecting pipe OB –

$$\frac{\partial^3 W(L,t)}{\partial x^3} = \frac{\partial^3 W(L,t)}{\partial x^2}\frac{1}{a} - \frac{P\pi\pi_m^{\ 2}}{EI}\frac{\partial W(1,t)}{\partial x} + \frac{k_b W(1,t)}{EI} + \frac{J * \partial 3\omega(L,t)}{EI * \partial x \partial t^2} + \frac{J_{Tot}\partial^2\omega(L,t)}{EI * a^2\,\partial t^2} \quad (8.11)$$

The total mass moment of inertia of fluid and metal of one-half of the connecting pipe about point of rotation 'O', including the inertial of lateral supports is expressed analytically as follows [1] –

$$J_{tot} = \frac{m_p + m_{ß}}{3} * a^3 + \frac{2m_p + m_{ß}}{4} * aR^2 + M_s a^2 \quad (8.12)$$

As in the lateral mode of analysis taking for 'w' in the expression (8.2), the boundary conditions (8.2), (8.3), (8.4) and (8.5) can be simplified by eliminating the time harmonic function and rewriting the expression as –

$X(0) = 0$ (8.13)

$$\frac{d^3 X(L)}{dx^3} = -\frac{1}{a}\frac{d^2 X(L)}{dx^2} - 2c^2 \frac{d\,X(L)}{dx} b\,X(L) \quad (8.14)$$

Where 'c' and 'λ' are defined in equations (7.5) & (7.6) respectively and b is given by $b = W^2 * \frac{J_{tot}}{EIa^2} - \frac{K_s}{EI}$ (8.15)

Using the equation (7.5) expression (8.14) is written as –

$\{\alpha^3 \cosh \alpha L + (\alpha^2/a) \sinh \alpha L + 2c^2 \cosh\alpha L + b \sinh \alpha L\}A + \{\alpha^3 \sinh \alpha L + (\alpha^2/a) \cosh \alpha L + 2c^2\, \alpha \sinh\alpha L + b \cosh \alpha L\}\, B - \{\beta^3 \cos\beta L + (\beta^2/a) \sin\beta L - 2c^2\beta \cos\beta L - b \sin\beta L\}C + \{\beta^3 \sin\beta L + (\beta^2/a) \cos\beta L - 2c^2\beta\sin\beta L + b \cos \beta L\}D = 0$ (8.16)

Grouping all the like terms together we get –

$\{\alpha(\alpha^2 + c^2) \cos h\alpha L + (\alpha^2/a+b) \sin h\alpha L\}\, A + \{\alpha(\alpha^2 + 2c^2) \sin h\alpha L + (\alpha^2/a + b) \cosh \beta L\}\, B - \{\beta(\beta^2 - 2c^2) \cos \beta L + (\beta^2/a-b) \sin \beta L\}\, C + \{\beta(\beta^2 - 2c^2) \sin\beta L - (\beta^2/a - b) \cos \beta L\}\, D = 0$ (8.17)

If

$a_1 = \alpha^2/a + b$ (8.18)

$a_2 = \alpha(\alpha^2 + 2c^2)$ (8.19)

$b_1 = \beta^2/a - b$ (8.20)

$b_2 = \beta(\beta^2 - 2c^2)$ (8.21)

$c_1 = a_1 \sinh \alpha L + a_2 \cosh \alpha L$ (8.22)

$c_2 = a_1 \cosh \alpha L + a_2 \sinh \alpha L$ (8.23)

$c_4 = b_2 \sin \beta L - b_1 \cos \beta L$ (8.24)

Then equation (8.17) can be written as follows –

$c_1 A + c_2 B - c_3 C + c_4 D = 0$ (8.25)

The determinant formed by the coefficients of this system of algebraic equations must be set equal to zero –

$$\begin{vmatrix} 0 & 1 & 0 & 1 \\ +T_1(\alpha L) & (\alpha L)^2 & +T_1(\beta L) & -(\beta L)^2 \\ d_1 & d_2 & d_3 & d_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix} \begin{vmatrix} A \\ B \\ C \\ D \end{vmatrix} = 0 \qquad (8.26)$$

Expanding the above matrix, we get the final frequency equation of double bellows in rocking mode–

$[-\alpha\beta ab_1 - \beta a_2 + \alpha\beta aa_1 + \alpha b_2\}$ $(1 - \cosh \alpha L \cos \beta L) - [-\beta_2 aa_1 - \beta b_2 + \alpha_2 ab_1 + \alpha a_2]$ $(\sin \beta L \sinh \alpha L) + [\alpha\beta ab_2 + \alpha\beta^2 a_2 + \alpha a_1 + \alpha b_1\}$ $\cosh \alpha L \sin\beta L + [\beta a_1 + \beta b_1 - a\alpha^2 b_2 + \alpha\beta a_2 a] \sinh \alpha L \cos \beta L = 0$ (8.27)

8.3 Derivation of Mode Shape Expression

At $X(0) = 0$

$$EI \frac{\partial x(0)}{\partial x^2} = R_1 \frac{\partial x(0)}{\partial x} \quad \text{at the end A}$$

$$\frac{\partial x(L)}{dx} = \frac{1}{a} x(L)$$

$$\frac{k_s}{EI} W(L,t) + \frac{J}{EI} \frac{\partial^2 W(L,T)}{\partial x^2 . \partial t^2} + \frac{J_{wt}}{EI \, . \, a^2} \cdot \frac{\partial^2 W(L,t)}{\partial t^2}$$

The above equation can be written as-

$$\frac{d^3(L)}{dx^3} = -\frac{1}{a} \frac{d^2x\,(L)}{dx^2} - 2C^2 \frac{d\,x(L)}{dx} - b\,x\,(L)$$

Where

$$\frac{P\pi Rm^2 + J\omega^2}{EI} = c$$

At X (0) = 0

$$X = A \sinh\alpha L + B \cosh\alpha L + C \sin\beta L + D \cos\beta L \qquad (8.28)$$

Applying X (0) = 0 in above equation, we get-

$$B + D = 0 \qquad (8.29)$$

$$\therefore \quad B = -D \qquad (8.30)$$

Now $\frac{d\,x(0)}{dx} = 0$

$$\therefore \quad \frac{dX}{dx} = A\alpha \operatorname{Cosh}\alpha L + B\beta \operatorname{Sinh}\alpha L + C\beta \operatorname{Cos}\beta L - D\beta \operatorname{Sin}\beta L \qquad (8.31)$$

$$\frac{d^2X}{dx^2} = A\alpha^2 \operatorname{Cosh}\alpha L + B\beta^2 \operatorname{Sinh}\alpha L + C\beta^2 \operatorname{Cos}\beta L - D\beta^2 \operatorname{Sin}\beta L \qquad (8.32)$$

$$EI \frac{d^2X(0)}{dx^2} = B\alpha^2 - D\beta^2 \qquad (8.33)$$

$$R_1 \frac{dX(0)}{dx} = (A\alpha + C\beta)\, R_1 \qquad (8.34)$$

Equating the above two equations, we get-

$$B\alpha^2 - D\beta^2 = \frac{R_1}{EI}(A\alpha + C\beta) \quad (8.35)$$

Multiplying by 'L' on both sides

$$B(\alpha L)^2 - D(\beta L)^2 = \frac{R_1 L}{EI}(A\alpha + C\beta) \quad (8.36)$$

$$B(\alpha L)^2 - D(\beta L)^2 = T_1(A\alpha + C\beta) \quad (8.37)$$

$$T_1 = \frac{R_1 L}{EI} \qquad \text{Substituting } B = -D \quad (8.38)$$

$$-D(\alpha L)^2 - D(\beta L)^2 = T_1(A\alpha + C\beta) \quad (8.39)$$

$$-D[(\alpha L)^2 - D(\beta L)^2] - T_1(A\alpha + C\beta) = 0$$

We know,

$$\frac{d\, X\,(L)}{d\,x} = A\alpha \cosh\alpha L + B\alpha \sinh\alpha L + C\beta \cos\beta L - D\beta \sin\beta L \quad (8.40)$$

And, $-\frac{1}{a}\,x\,(L) = -\frac{1}{a}[A \sinh \alpha L + B \cosh \alpha L + C \sin \beta L + D \cos \beta L]$

(8.41)

$\therefore$ $a[A\alpha \operatorname{Cosh} \alpha L + B\alpha \operatorname{Sinh}\alpha L + C\beta \operatorname{Cosh} \alpha L + D \operatorname{Sin} \beta L] + [A \operatorname{Sinh} \alpha L +$
$B \operatorname{Cosh} \alpha L + C \operatorname{Sin} \beta L + D \operatorname{Cos}\beta L] = 0$ (8.42)

$A[\alpha ac\operatorname{Cosh} \alpha L + \sinh \alpha L] + B[\alpha a \sinh \alpha L + \cosh \alpha L] + C[a\beta \cos\beta L + \sin \beta L] -$
$D[a\beta \sin \beta L - \cos \beta L)] = 0$ (8.43)

Writing B = – D,

$$A\,[\alpha a \cosh \alpha L + \sinh \alpha L] + D\,[\alpha a \sinh \alpha L + \cosh \alpha L] + C\,[a\beta \cos\beta L + \sin \beta L] -$$

$$D\,[a\beta \sin \beta L - \cos \beta L)] = 0 \qquad (8.44)$$

$$= A\,[\alpha a \cosh \alpha L + \sinh \alpha L] + C\,[a\beta \cos \beta L + \sin \beta L] + D\,[\alpha a \sin \alpha L + \cosh\alpha L + a\beta \sin \beta L - \cos \beta L] = 0 \qquad (8.45)$$

We know, $-T_1 (A\alpha + C\beta) = D\,(\alpha L)^2 + (\beta L)^2]$

$$A\,[\alpha a \cosh \alpha L + \sinh \alpha L] + C\,[a\beta \cos \beta L + \sin \beta L] = D[\alpha a \sin \alpha L + \cosh \alpha L + a\beta \sin \beta L - \cos \beta L] \qquad (8.46)$$

Multiplying LHS by $(a\alpha \cosh \alpha L + \sinh \alpha L)$ and RHS by T_1Aa we get-

$$- T_1A\,(a\alpha \cosh \alpha L + \sinh \alpha L) - T_1B\,(a\alpha \cosh \alpha L + \sinh \alpha L) = D\,(a\alpha \cosh \alpha L + \sinh \alpha L)\,[(\alpha L)^2 + (\beta L)^2]$$

$$T_1Aa\,(a\alpha \cosh \alpha L + \sinh \alpha L) + C\,T_1a\,(a\beta \cos \beta L + \sin \beta L) = DT_1a\,(\alpha a \sinh \alpha L + \cosh \alpha L + a\beta \sin\beta L - \cos \beta L]$$

Canceling 'A' terms and adding, we get-

$$- T_1C\beta(a\alpha \cosh \alpha L + \sinh \alpha L) + C\,T_1a\,(a\beta \cos \beta L + \sin \beta L) = D\,[a\alpha \cosh \alpha L + \sinh \alpha L)\,[(\alpha L)^2 + (\beta L)^2] - T_1a(\alpha a \sinh \alpha L + \cosh \alpha L + a\beta \sin\beta L - \cos \beta L] \qquad (8.47)$$

$C\{-T_1\beta(a\alpha \cosh \alpha L + \sinh \alpha L) + T_1 a (a\beta \cos \beta L + \sin \beta L) = D[a\alpha \cosh \alpha L + \sinh \alpha L) [(\alpha L)^2 + (\beta L)^2] - T_1 a(\alpha a \sinh \alpha L + \cosh \alpha L + a\beta \sin\beta L - \cos \beta L]\}$

(8.48)

$$\frac{C}{D} = \left\{ \frac{T_1\beta(a\alpha \operatorname{Cosh} \alpha L + \operatorname{Sinh} \alpha L) + T_1 a\ (a\beta \operatorname{Cos} \beta L + \operatorname{Sin} \beta L)}{(a\alpha \operatorname{Cosh} \alpha L + \operatorname{Sinh} \alpha L)[(\alpha L)^2 + (\beta L)^2 - T_1 a[\alpha a \operatorname{Sinh} \alpha L + \operatorname{Cosh} \alpha L + a\beta \operatorname{Sin} \beta L - \operatorname{Cos} \beta L]} \right\}$$

(8.51)

Now canceling 'C' terms to obtain the ratio A/D-

$T_1 A\alpha(a\beta \cos \beta L + \sin \beta L) + C\beta (a\beta \cos \beta L + \sin \beta L) = D [a\alpha \cos \beta L + \sin \beta L) (\alpha a \sinh \alpha + \cosh \alpha L + a\beta \sin\beta L - \cos \beta L]$

$A\beta(a\alpha \cosh \beta L + \sinh \alpha L) + C\beta (a\beta \cos \beta L + \sin \beta L) = D\beta [a\alpha \sinh \alpha L + \cosh \alpha L + a\beta \sin\beta L - \cos \beta L]$

Canceling 'C' terms we get,

$T_1 A\alpha(a\beta \cos \beta L + \sin \beta L) + A\beta (a\beta \cosh \alpha L - \sinh \alpha L) = D [a\beta \cos \beta L + \sin \beta L) (\alpha a \sinh \alpha + \cosh \alpha L + a\beta \sin\beta L - \cos \beta L] + \beta[a\alpha \sinh \alpha L + \cosh \alpha L + a\beta \sin\beta L - \cos \beta L]$ (8.52)

$A [-T_1\alpha(a\beta \operatorname{Cos} \beta L + \operatorname{Sin} \beta L) + \beta (a\alpha \operatorname{Cosh} \alpha L - \operatorname{Sinh} \alpha L) = -D [\alpha\beta a^2 \sinh \alpha \cos \beta L + a\beta \cos \beta L \cosh \alpha L + a^2 \beta^2 \cos \beta L \sin \beta L - a\beta \cos^2 \beta L + \alpha a \sin \beta L \sinh \alpha L + \cosh \alpha L \sin \beta L + a\beta \sin^2 \beta L - \cos \beta L \sin \beta L] + \beta [(\alpha a \sinh \alpha L + \cosh \alpha L\ a\beta \sin \beta L - \cos \beta L)]$ (8.53)

$= \quad D\{[\alpha\beta a^2 \sinh \alpha \cos \beta L + a\beta \cos \beta L \cosh \alpha L + \cos \beta L \sin \beta L(a^2-1) - a\beta (\cos^2 \beta L + \sin^2 \beta L) + \alpha a \sin \beta L \sinh \alpha L + \cosh \alpha L \sin \beta L] + \beta (\alpha a \sinh \alpha L + \cosh \alpha L + a\beta \sin \beta L - \cos \beta L]\}$

$$\therefore \frac{A}{D} = \left\{ \frac{T_1\alpha\,(a\beta \operatorname{Cosh}\beta L + \operatorname{Sin}\,\beta L) + \beta(\alpha a \operatorname{Cosh}\,\alpha L - \operatorname{Sinh}\,\alpha L)}{\begin{array}{l}(\alpha\beta a^2 \operatorname{Sinh}\,\alpha L\, \operatorname{Cosh}\beta L + a\beta \operatorname{Cos}\beta L\, \operatorname{Cosh}\alpha L + \operatorname{Cos}\,\beta L \operatorname{Sin}\,\beta L(a^2-1) \\ -a\beta + \alpha a\, \operatorname{Sin}\,\beta\, \operatorname{Sinh}\,\alpha L + \operatorname{Cosh}\alpha L \operatorname{Sin}\,\beta L] + \beta[\alpha a\, \operatorname{Sinh}\,\alpha L + \operatorname{Cosh}\alpha L + \\ a\beta \operatorname{Sin}\beta L - \operatorname{Cos}\beta L]\end{array}} \right\}$$

$$A' = \frac{A}{D}$$
$$C' = \frac{C}{D}$$

$$\frac{B}{D} = -1$$

The final mode shape expression can be written as-

$$\therefore \frac{y(\xi)}{D} = A'\sinh\,\alpha L - B\cosh\alpha L + C'\sin\,\beta L + D\cos\beta L$$

8.4 Results and Discussion

The above frequency solutions for rocking mode are tested by considering a bellow with the following geometrical and physical parameters –

Bellows length, L = 0.00693 m, mass moment of inertia per unit length, J = 0.001153kgm, EI = 5.078Nm^2, total bellows mass, m_{tot} = 5.138kg/m and total connecting pipe mass, $m_p + m_{f3}$ = 5.0kg/m.

The study involves the effect of varying the equivalent support stiffness and mass on the frequency of vibration. Using expression (7.11), (7.12) and (8.12), (8.13), (8.14), (8.15) the frequency equation (8.27) is solved using bisection method. The first fundamental natural frequencies obtained for rocking modes are given in Tables 8.1 & 8.2 by varying spring stiffness k_s and support mass M_s respectively.

It is seen that as stiffness increases from a value of 10^2 to 10^{15}, frequency increases from 936.10rad/s to 3856.1719rad/s, a rise by almost 75%. The frequency attains almost a constant value form 10^8 onwards. The effect of varying the support mass on the frequency of vibration is also studied and the results presented in Table 8.2. It is seen that there is a steep drop of 87% in frequency as the support mass increases form 1.0 to 100 for N = 1 and by a small drop of 2% for N = 2. The exact frequencies obtained by the bisection method are compared with the results presented in the thesis [1] and are presented in Table 8.3.

It is seen that the percentage error obtained in the frequency of vibration for a bellow with a lateral support having a stiffness of $k_s = 10^2$ quite small and comparable to a bellow with no lateral supports.

Table 8.1 Rocking Natural Frequency for N=1,2 T=∝ & M_s=0

Spring Support Stiffness, k_s	**Exact, rad/s N1**	**Exact, rad/s N2**
10^2	936.1077	3856.172
10^3	937.5491	3856.275
10^4	951.8327	3857.306
10^5	1083.4352	3867.815
10^6	1904.5623	3993.815
10^7	3232.5317	5950.792
10^8	3386.146	6849.09
10^9	3390.1234	6886.407
10^{10}	3400.0227	6889.807

Fig 8.2 Exact Rocking Natural Frequency -Double Bellows
For N=1 &2 and M_s=0

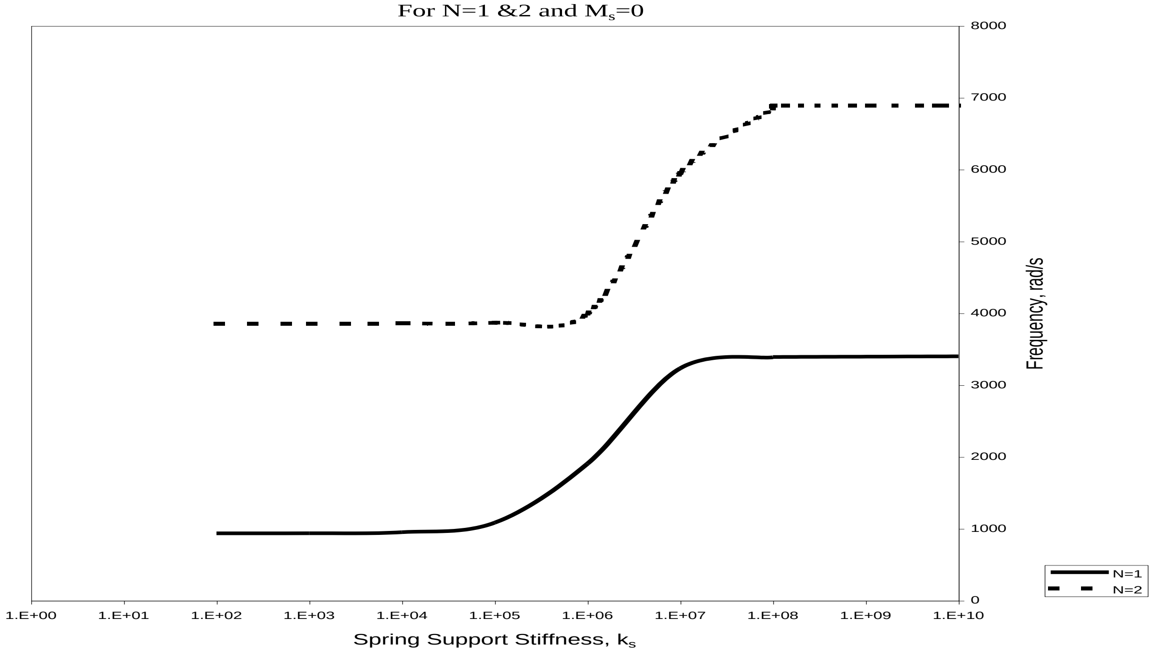


Table 8.2 Rocking Natural Frequency for N=1, 2 and T=∝ & k_s=0

Support Mass ms	Exact, rad/s N=1	Exact, rad/s N=2
1	429.1701	3480.089
10	164.6297	3411.481
20	117.9629	3405.954
30	96.75	3404.053
40	83.977	3403.092
50	75.2143	3402.511
60	68.7234	3402.122
70	63.667	3401.844
80	59.584	3401.635
90	56.197	3401.472
100	53.33	3401.341

Fig 8.3 Exact Rocking Mode Natural Frequency -Double Bellows
For N=1,2 and k_s=0

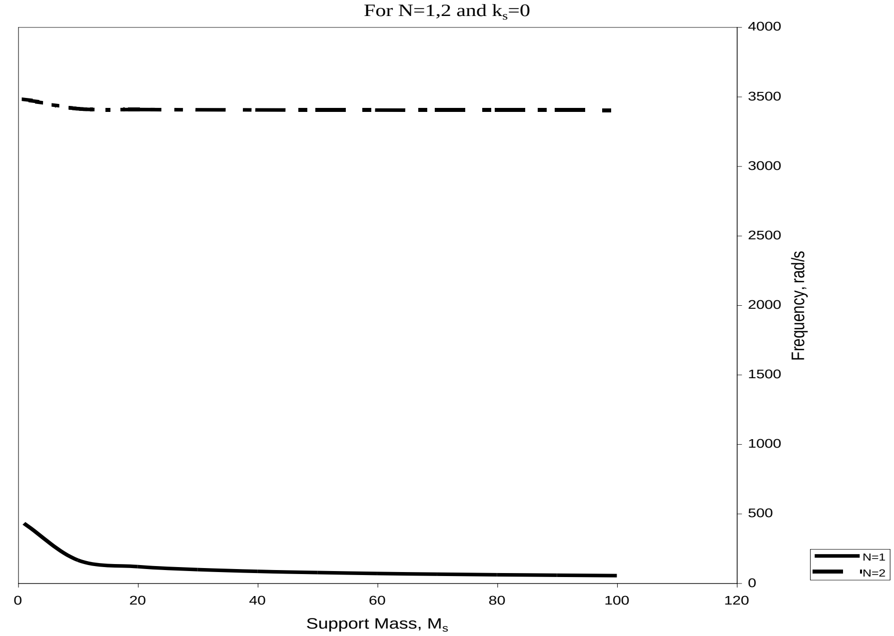
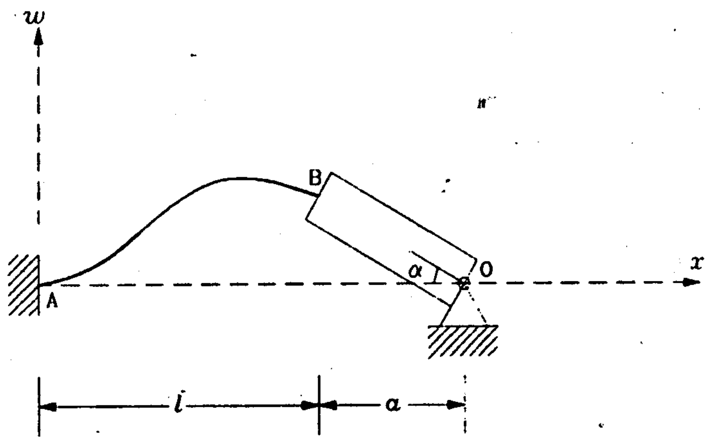


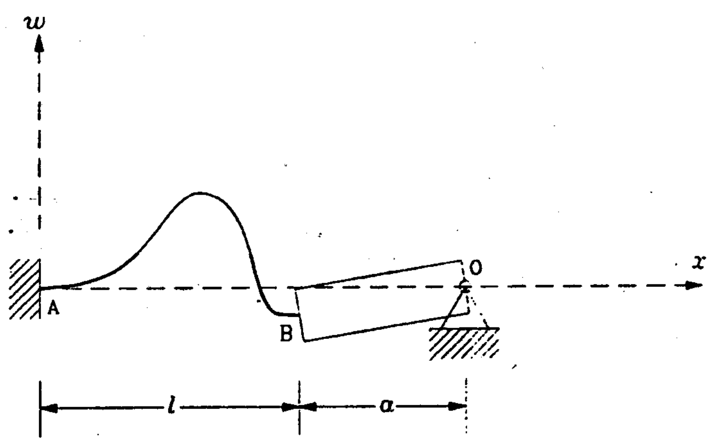
Table 8.3 Comparison of frequencies at T=∝

Mode N	[28] ω, rad/s $k_s=10^2$	Exact [46]ω, rad/s $k_s = 10^2$	Percent Error
1	935.686	936.107	0.01
2	3855.928	3856.1719	0.006

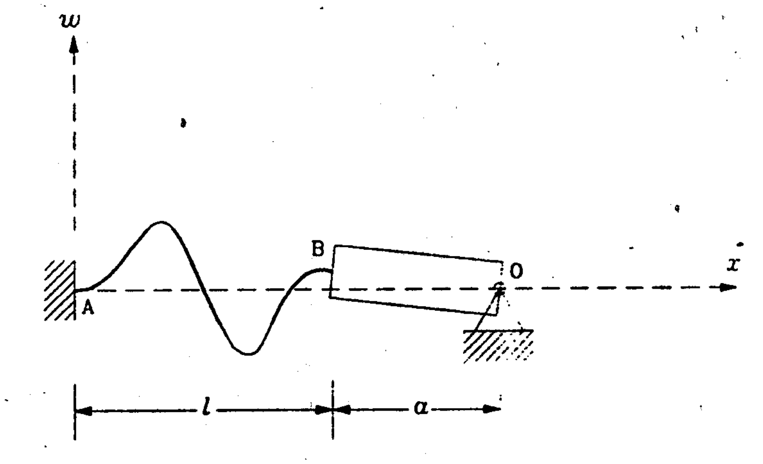
Figures 8.4a,b&c represent the first three mode shapes of bellows in rocking mode.



a) First mode shape



b) Second mode shape



c) Third mode shape

Fig 8.5 First Mode Shape of Double Bellows with Elastically Restrained end -(Rocking Mode) Support Mass, M_s=0 & N=1, T=infinity

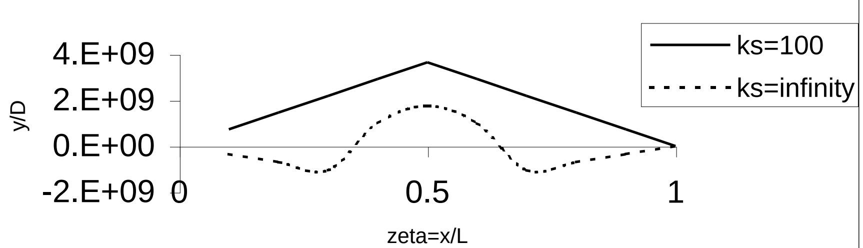
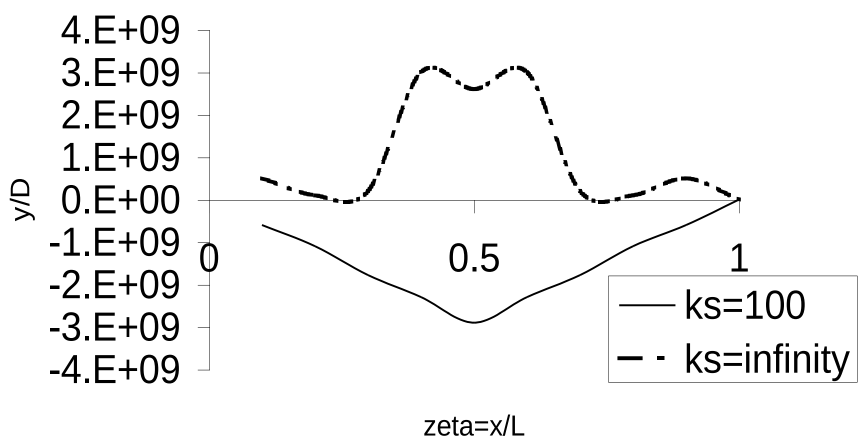


Fig 8.6 Second Mode Shape of Double Bellows with Elastically Restrained End, Support Mass M_s=0, T=infinity & N=2



The first and second mode shapes for the double bellows in rocking mode are shown in Figures 8.5and 8.6 respectively. By varying ξ between 0 to 1.0 for the full cycle and keeping rotational restraint T =∝, the response is observed for support stiffness k_s=100 and k_s=∝. It is seen that as the support stiffness ks, increases the response of bellows changes signs. This should be a good indicator of controlling the bellows vibration by altering the stiffness.

CHAPTER 9

SEISMIC RESPONSE OF ELASTICALLY RESTRAINED SINGLE BELLOWS EXPANSION JOINT IN LATERAL MODE

9.1 Background

It is seen that most of the nuclear and thermal power plants around the world are located in the seismic zones. The piping in these plants contains expansion bellows and needs to be analyzed from seismic design point of view. Hence, while designing an expansion bellow from structural point of view, the seismic design aspect is also to be given utmost importance.

The present work therefore attempts to derive an exact solution for the seismic response of U type of single bellows that are considered elastically restrained against rotation to classical fixed-fixed case considered by Morishita et al. Hence, it is observed that this assumption will lead to an over estimation of natural frequencies and incorrect determination of seismic response.

The fundamental equation of motion of the equivalent Euler-Bernoulli beam is used to calculate the spectral response of the bellows to a seismic excitation in the lateral direction. The present study excludes shear and inertia terms of the equivalent Timoshenko beam for the bellows. The first lateral mode frequencies are obtained for the case of equal rotational spring stiffness on either end.

9.2 Simplified Analysis of Lateral Mode of Vibration

Consider a uniform Euler-Bernoulli beam with its both ends rotationally restrained which has the equivalent dynamic characteristics as a bellows and is excited in lateral direction as shown in Fig 9.1. Here we use the exact method to obtain the natural frequencies. Let the deflection of the beam by bending be y (z, t) and assuming sinusoidal vibration-

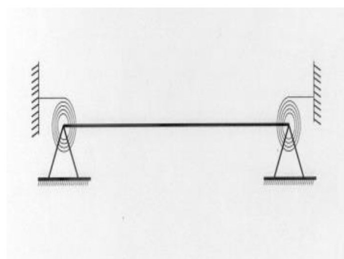


Fig 9.1 Single Bellows Restrained Against Rotation

$$y(z,t) = Y(z)\, e^{i\omega t} \qquad (9.1)$$

Where 'ω' is the natural frequency of vibration, t is time in seconds, Z= z/L, the non-dimensional length of bellows and Y (z) is given by-

$$Y(z) = A \sinh \alpha z + B \cosh \alpha z + C \sin \alpha z + D \cos \alpha z \qquad (9.2)$$

Where A, B, C and D are arbitrary constants

Equation (9.2) is considered as the solution to the differential equation –

$$\frac{d^4 y}{dz^4} - \lambda^4 y = 0 \tag{9.3}$$

Where

$$\lambda^4 = \frac{m_{tot}\,\omega^2}{EI} \tag{9.4}$$

Where EI is bending stiffness, R_m –mean radius of bellows, m_{tot} is the total mass of bellows per unit length that includes mass of bellows and fluid mass [40].

The physical quantities of the beam such as EI is given in terms of the dimensions and material constants of the bellows and are as follows-

$$EI = \frac{5}{24}\frac{(d_p.t_p)^3 qE}{h^3.C_f} \tag{9.5}$$

$$\rho A = 2\pi\rho_b\left(\frac{t.L}{p} + \rho_f \frac{dp}{8}\right)d_p \tag{9.6}$$

Where d_p is the pitch diameter of bellows, t_p is the wall thickness of bellows, h is height of convolution, $\rho_{(b,\,f)}$ density of bellows material and fluid and q is the pitch.

The frequency equation is derived and the first fundamental transverse frequency is obtained using bisection method.

9.4 Boundary Conditions

The boundary conditions for the rotationally restrained Euler-Bernoulli beam / bellow are as follows-

i) $y = 0$ at $z=0$ (9.7)

ii) $\partial^2 y / \partial z^2 = + T_1 . \partial y/\partial z$ at $z=0$ (9.8)

iii) And $y = 0$ at $z = L$ (9.9)

iv) $\partial^2 y / \partial z^2 = - T_2 \, \partial y/\partial z$ (9.10)

Let the solution for the differential equation (9.3) be written as-

$$y(z) = A \text{ Sinh } \alpha z + B \text{Cosh} \alpha z + C \text{ Sin } \alpha z + D \text{ Cos } \alpha z \tag{9.11}$$

$$\frac{dy}{dz} = \alpha \left(A \text{ Cosh } \alpha z + B \text{ Sinh } \alpha z + C \text{ Cos } \alpha z - D \text{ Sin } \alpha z\right) \tag{9.12}$$

$$\frac{d^2 y}{dz^2} = \alpha^2 \left(A \text{ Sinh } \alpha z + B \text{ Cosh } \alpha z - C \text{ Sin } \alpha z - D \text{ Cos } \alpha z\right) \tag{9.13}$$

$$\text{(i)} \quad y = 0 \text{ at } z = 0; \; B + D = 0 \tag{9.14}$$

$$\text{(ii)} \; \frac{d^2 y}{dz^2} = + \; T_1 \; \frac{dy}{dz}; \quad \text{at } z = 0 \tag{9.15}$$

$$\left(\alpha L^2\right)(B - D) = + \; T_1 \; \alpha (A + C) \tag{9.16}$$

$$B + D = 0 \;\; \therefore D = -B$$

$$\therefore \alpha^2(B+B) + T_1\ \alpha\ (A+C)$$

$$2(\alpha L)(BL) = T_1(A+C)$$

$$\therefore\ B = \frac{T_1}{2(\alpha L)}(A+C) \qquad (9.17)$$

$$D = -B = -\frac{T_1}{2(\alpha L)}(A+C) \qquad (9.18)$$

(iii) $y = 0$ at $z = 1$

$$A\ \mathrm{Sin}\,\alpha L + B\ \mathrm{Cosh}\ \alpha L\ + C\ \mathrm{Sin}\ \alpha L + D\ \mathrm{Cos}\,\alpha L \qquad (9.19)$$

$(\alpha L)^2$ (A sin hαL+ B cos hαL - C sin αL - D cos αL) =

$$-T_2\ (\alpha L)(A\ \mathrm{Cosh}\ \alpha L + B\ \mathrm{Sinh}\ \alpha L + C\ \mathrm{Cos}\ \alpha L - D\ \mathrm{Sin}\,\alpha L) \qquad (9.20)$$

$$\therefore\ (\alpha L)(A\,\mathrm{Sinh}\ \ \alpha L + B\ \mathrm{Cosh}\ \alpha L - D\ \mathrm{Cos}\ \alpha L)$$
$$+\ T_2\ (A\ \mathrm{Cosh}\ \alpha L + B\ \mathrm{Sinh}\ \ \alpha L + C\ \mathrm{Cos}\ \alpha L -\ D\ \mathrm{Sin}\ \ \alpha L) = 0$$

$$(9.21)$$

We know-

$$A\,\mathrm{Sinh}\ \ \alpha L + B\ \mathrm{Cosh}\ \alpha L +\ C\ \mathrm{Sinh}\ \ \alpha L + D\ \mathrm{Cos}\ \alpha L\ = 0$$

$$(\alpha L(A) + T_2(BL))\mathrm{Sinh}\ \ \alpha L + (\alpha LB + T_2A)\ \mathrm{Cosh}\ \alpha L - (\alpha LC + T_2D)\ \mathrm{Sin}\,\alpha L$$

$$-(\alpha LD - T_2c)\mathrm{Cos}\,\alpha L = 0 \qquad (9.22)$$

$$A\ \mathrm{Sinh}\ \ \alpha L + B\ \mathrm{Cosh}\ \alpha L + C\ \mathrm{Sin}\ \ \alpha L - B\ \mathrm{Cos}\ \alpha L = 0$$

$$\therefore\ A\ \mathrm{Sinh}\ \ \alpha L + C\ \mathrm{Sin}\ \ \alpha L +\ B\ (\mathrm{Cosh}\ \alpha L - \mathrm{Cos}\ \alpha L) = 0 \qquad (9.23)$$

$$(\alpha LA+T_2B)\,Sinh\,\alpha L+(\alpha B+T_2A)Cosh\,\alpha L$$

$$-(\alpha LC+T_2B)\,Sin\,\alpha L-(-\alpha LB+T_2C)\,Cos\,\alpha L=0 \tag{9.24}$$

$$(\alpha LA+T_2B)\,Sinh\,\alpha L+(\alpha B+T_2A)Cosh\,\alpha L$$

$$-(\alpha LC-T_2)\,Sin\,\alpha L+(\alpha LB+T_2C)\,Cos\,\alpha L=0 \tag{9.25}$$

$$B=-\frac{(A\ Sinh\ \alpha L+C\ Sin\ \alpha L)}{(Cosh\ \alpha L-Cos\ \alpha L)}$$

$$B\{T_2\ Sinh\ \alpha L+\alpha\ Cosh\ \alpha L+T_2\ Sin\ \alpha L+\alpha\,Cos\alpha L\}$$

$$+A\{\alpha\ Sinh\ \alpha L+T_2\,Cosh\,\alpha L\}+C\{T_2\ Cos\alpha L-\alpha\,Sin\alpha L\}=0 \tag{9.26}$$

$$B=-\left(\frac{A\ Sinh\ \alpha L+C\ Sin\ \alpha L}{Cosh\,\alpha L-Cos\ \alpha L}\right) \tag{9.27}$$

$$B=\left(\frac{\pi}{2\alpha L}\right)(A+C) \tag{9.28}$$

$$\therefore\left(\frac{\pi}{2\alpha}\right)(A+C)=-\left\{\frac{A\ Sinh\,\alpha L+C\ Sin\ \alpha L}{(Cosh\ \alpha L-Cos\ \alpha L)}\right\} \tag{9.29}$$

$$\therefore\pi(A+C)\,(Cosh\alpha L-Cos\alpha L)=-2\alpha(A Sinh\ \alpha L+C Sin\ \alpha L)$$

$$\therefore A\{T_1(Cosh\,\alpha L-Cos\ \alpha L)+2\alpha Sinh\,\alpha L\}$$

$$=-\{T_1(Cosh\,\alpha L-Cos\ \alpha L)+2\alpha Sinh\,\alpha L\}\,C$$

$$\therefore\frac{C}{A}=-\left[\frac{T_1(Cosh\alpha L-Cos\alpha L)+2\alpha\,Sinh\,\alpha L}{T_1(Cosh\alpha L-Cos\alpha L)+2\alpha\,Sin\,\alpha L}\right] \tag{9.30}$$

Substituting for B in last equation – we get

$$\frac{T_1}{2\alpha}(A+C)\{T_2\ Sinh\,\alpha L+\alpha L\,Cosh\,\alpha L+T_2\ Sin\,\alpha L+\alpha L\,Cos\alpha L\}$$

$$+A(\alpha L Sinh\ \alpha L+T_2\,Cosh\ \alpha L)+C(T_2\ Cos\ \alpha L-\alpha\,Sin\alpha L)=0 \tag{9.31}$$

$$\therefore T_1(A + C)(\alpha \operatorname{Cosh} \alpha L + T_2 \sinh \alpha L + \alpha \cos \alpha L + T_2 \sin \alpha L)$$

$$+ 2 \alpha A (\alpha \sinh \alpha L + T_2 \cosh \alpha L) + 2\alpha c(T_2 \cos \alpha L - \alpha \sin \alpha L) = 0 \qquad (9.32)$$

$$A\{T_1(\alpha \cosh \alpha L + T_2 \sinh \alpha L + \alpha \cos \alpha L + T_2 \sin \alpha L) + 2\alpha(\alpha \sinh \alpha L + T_2 \cosh \alpha L)\}$$

$$- C\{T_1(\alpha \cosh \alpha L + T_2 \sinh \alpha L + \alpha \cos \alpha L + T_2 \sin \alpha L) - 2\alpha (\alpha \sin \alpha L - T_2 \operatorname{Cos} \alpha L)\}$$

$$\frac{C}{A} = -\left[\frac{T_1(\alpha \cosh \alpha L + T_2 \sinh \alpha L + \alpha \cos \alpha L + T_2 \sin \alpha L) + 2\alpha(\alpha \sinh \alpha L + T_2 \cosh \alpha L)}{T_1(\alpha \cosh \alpha L + T_2 \sinh \alpha L + \alpha \cos \alpha L + T_2 \sin \alpha L) - 2\alpha (\alpha \sin \alpha L - T_2 \operatorname{Cos} \alpha L)}\right]$$

$$(9.33)$$

Equating the two expressions for $\frac{C}{A}$ we obtain-

$$-\left[\frac{T_1(\cosh \alpha L - \cos \alpha L) + 2\alpha \sinh L\alpha}{T_1(\cosh \alpha L - \cos \alpha L) + 2\alpha \sin \alpha L}\right]$$

$$= -\left[\frac{T_1(\alpha \cosh \alpha L + T_2 \sinh \alpha L + \alpha \cos \alpha L + T_2 \sin \alpha L) + 2\alpha(\alpha \sinh \alpha L + T_2 \cosh \alpha L)}{T_1(\alpha \cosh \alpha L + T_2 \sinh \alpha L + \alpha \cos \alpha L + T_2 \sin \alpha L) - 2\alpha(\alpha \sinh \alpha L - T_2 \cosh \alpha L)}\right]$$

$$(9.34)$$

By cross-multiplying we get-

$$(T_1(\cosh \alpha L - \cos \alpha L) + 2\alpha \sin \alpha L)\begin{bmatrix} T_1(\alpha \cosh \alpha L + T_2 \sinh \alpha L + \alpha \cos \alpha L + T_2 \sin \alpha L) \\ + 2\alpha(\alpha \sinh \alpha L + T_2 \cosh \alpha L) \end{bmatrix}$$

$$= (T_1(\cosh \alpha L - \cos \alpha L) + 2\alpha \sin \alpha L)\begin{bmatrix} T_1(\alpha \cosh \alpha L + T_2 \sinh \alpha L + \alpha \cos \alpha L + T_2 \sin \alpha L) \\ - 2\alpha(\alpha \sin \alpha L - T_2 \cos \alpha L) \end{bmatrix}$$

$$(9.35)$$

$$\left(T_1^2(\cosh \alpha L - \cos \alpha L) + \begin{pmatrix} \alpha \cosh \alpha L + T_2 \sinh \alpha L + \alpha \cos \alpha L + T_2 \sin \alpha L + T_1 \alpha(\cosh \alpha L - \cos \alpha L) \\ (\alpha \sinh \alpha L + T_2 \cosh \alpha L \end{pmatrix}\right.$$

$$+2T_1\,\alpha\sin\alpha\left(\alpha\cosh\alpha+T_2\sinh\alpha+\alpha\cos\alpha+T_2\sin\alpha\right)+4\alpha^2\sin\alpha\left(\alpha\sinh\alpha+T_2\cosh\alpha\right)\Big)$$

(9.36)

$$\left(T_1^{\,2}\left(\cosh\alpha-\cos\alpha\right)+\left(\begin{matrix}(\alpha\cosh\alpha+T_2\sinh\alpha+\alpha\cos\alpha+T_2\sin\alpha)\;+\\ 2\,T_1\,\alpha\left(\cosh\alpha-\cos\alpha\right)\left(\alpha\sinh\alpha-T_2\cosh\alpha\right.\end{matrix}\right)\right.$$

$$+2T_1\,\alpha\sinh\alpha\left(\alpha\cosh\alpha+T_2\sinh\alpha+\alpha\cos\alpha+T_2\sin\alpha\right)-4\alpha^2\sinh\alpha\left(\alpha\sin\alpha-T_2\cosh\alpha\right)\Big)$$

(9.37)

We get-

$$2T_1\alpha\left(\cosh\alpha-\cos\alpha\right)\left(\alpha\sinh\alpha+T_2\cosh\alpha+\alpha\sin\alpha-T_2\cos\alpha\right)$$
$$+2T_1\alpha\left(\alpha\cosh\alpha+T_2\sinh\alpha+\alpha\cos\alpha+T_2\sin\alpha\right)\left(\sin\alpha-\sinh\alpha\right)$$
$$+4\alpha^2\left(\alpha\sinh\alpha.\sin\alpha+T_2\mathrm{consh}\alpha\sin\alpha+\alpha\sinh\alpha\sin\alpha-T_2\sinh\alpha\cos\alpha\right)=0$$

(9.38)

$$2\alpha\left[2\alpha\sinh\alpha\sin\alpha+T_2\left(\cosh\alpha\sin\alpha-\sinh\alpha\cos\alpha\right)\right]$$
$$+T_1\left(\sin\alpha-\sinh\alpha\right)\left[\alpha\left(\cosh\alpha+\cos\alpha\right)+T_2\left(\sinh\alpha+\sin\alpha\right)\right]$$
$$+T_1\left(\cosh\alpha-\cos\alpha\right)\left[\alpha\left(\sinh\alpha+\sin\alpha\right)+T_2\left(\cosh\alpha-\cos\alpha\right)\right]=0$$

(9.39)

$$2\alpha\left[2\alpha\sinh\alpha\sin\alpha+T_2\left(\cosh\alpha\sin\alpha-\sinh\alpha\cos\alpha\right)\right]$$
$$-T_1\left(\sinh\alpha-\sin\alpha\right)\left[\alpha\left(\cosh\alpha+\cos\alpha\right)+T_2\left(\sinh\alpha+\sin\alpha\right)\right]$$
$$+T_1\left(\cosh\alpha-\cos\alpha\right)\left[\alpha\left(\sinh\alpha+\sin\alpha\right)+T_2\left(\cosh\alpha-\cos\alpha\right)\right]=0$$

(9.40)

$$4\alpha^2 \sinh\alpha \sin\alpha + 2T_2(\cosh\alpha \sin\alpha - \sinh\alpha \cos\alpha)$$

$$+ T_1\alpha(\sinh\alpha \cosh\alpha - \sinh\alpha \cos\alpha + \sin\alpha \cosh\alpha - \cos\alpha \sin\alpha$$

$$- \sinh\alpha \cosh\alpha - \sinh\alpha \cos\alpha + \sin\alpha \cosh\alpha + \cos\alpha \sin\alpha)$$

$$+ T_1T_2\left[\cosh^2\alpha + \cos^2\alpha - 2\cos\alpha \cosh\alpha - \sinh^2\alpha + \sin^2\alpha\right] = 0 \qquad (9.41)$$

$$T_1T_2\left[\left(\cosh^2\alpha - \sinh^2\alpha\right) + \left(\cos^2\alpha + \sin^2\alpha\right) - 2\cos\alpha \cosh\alpha\right]$$

$$+ 2T_1\alpha\left[\sin\alpha \cosh\alpha - \cos\alpha \sinh\alpha\right]$$

$$+ 2T_2\alpha(\sin\alpha \cosh\alpha - \cos\alpha \sinh\alpha) + 4\alpha^2 \sinh\alpha \sin\alpha = 0$$

(9.42)

i.e., $2T_1T_2(1 - \cos\alpha \cosh\alpha) + 2\alpha(T_1 + T_2)(\sin\alpha \cosh\alpha - \cos\alpha \sinh\alpha)$
$+ 4\alpha^2 \sinh\alpha \sin\alpha = 0$

(9.43)

$$2\alpha^2 \sinh\alpha \sin\alpha + \alpha\,(T_1 + T_2)(\sin\alpha \cosh\alpha - \cos\alpha \sinh\alpha) + T_1T_2(1 - \cos\alpha \cosh\alpha) = 0$$

(9.44)

$$= \bar{c}\left\{\sin\alpha z - \left[\frac{2\alpha \sin\alpha + T_1(\cosh\alpha - \cos\alpha)}{2\alpha \sinh\alpha + T_1(\cosh\alpha - \cos\alpha)}\right]\sinh\alpha z\right.$$

$$\left. + \frac{T_1}{2\alpha}(\cosh\alpha - \cos\alpha z)\left[1 - \frac{2\alpha \sin\alpha + T_1(\cosh\alpha - \cos\alpha)}{2\alpha \sinh\alpha + T_1(\cosh\alpha - \cos\alpha)}\right]\right.$$

(9.45)

$$= \bar{c}\left\{\sin\alpha z - \left[\frac{2\alpha \sin\alpha + T_1(\cosh\alpha - \cos\alpha)}{2\alpha \sinh\alpha + T_1(\cosh\alpha - \cos\alpha)}\right]\sinh\alpha z\right.$$

$$+ \frac{T_1}{2\alpha}\left[\frac{2\alpha \sinh\alpha + T_1(\cosh\alpha - \cos\alpha) - 2\alpha \sin\alpha - T_1(\cosh\alpha - \cos\alpha)}{2\alpha \sinh\alpha + T_1(\cosh\alpha - \cos\alpha)}\right]$$

$$\left. .\,(\cosh\alpha z - \cos\alpha z)\right\} \qquad (9.46)$$

$$= \bar{c}\left\{\sin\alpha z - \left[\frac{2\alpha\sin\alpha + T_1(\cosh\alpha - \cos\alpha)}{2\alpha\sinh\alpha + T_1(\cosh\alpha - \cos\alpha)}\right]\sinh\alpha z\right.$$

$$\left. + \frac{T_1}{2\alpha}\left[\frac{2\alpha(\sinh\alpha - \sin\alpha)}{2\alpha\sinh\alpha + T_1(\cosh\alpha - \cos\alpha)}\right](\cosh\alpha z - \cos\alpha z)\right\} \tag{9.47}$$

$$y(z) = \bar{c}\left\{\sin\alpha z - \left[\frac{2\alpha\sin\alpha + T_1(\cosh\alpha - \cos\alpha)}{2\alpha\sinh\alpha + T_1(\cosh\alpha - \cos\alpha)}\right]\sinh\alpha z\right.$$

$$\left. - \left[\frac{T_1(\sinh\alpha - \sin\alpha)}{2\alpha\sinh\alpha + T_1(\cosh\alpha - \cos\alpha)}\right](\cos\alpha z - \cosh\alpha)\right\} \tag{9.48}$$

The final frequency equation is -

$$2(\alpha L^2)\sin\alpha L\,\sinh\alpha L + (\alpha L)(T_1 + T_2)(\sin\alpha L\,\cosh\alpha L - \cos\alpha L\,\sinh\alpha L)$$

$$+ T_1T_2(1 - \cos\alpha L\,\cosh\alpha L) = 0 \tag{9.49}$$

The expression for mode shape is as follows-

$$y(z) = A\sinh\alpha + B\cosh\alpha z + C\sin\alpha z + D\cos\alpha z$$

We have B+D=0 & $B = \frac{T_1}{2\alpha}(A + C)$

$$\therefore y(z) = A\sinh(\alpha L)z + C\sin(\alpha L)z + B(\cosh(\alpha L)z - \cosh(\alpha L)z)$$

$$= A\sinh(\alpha L)z + C\sin(\alpha L)z + \frac{T_1}{2\alpha}(A + C)(\cosh(\alpha L)z - \cosh(\alpha L)z) \tag{9.50}$$

$$= A\left[\sinh\alpha Lz + \frac{T_1}{2\alpha}(\cosh\alpha Lz - \cos\alpha Lz)\right] + C\left[\sin\alpha Lz + \frac{T_1}{2\alpha}(\cosh\alpha Lz - \cos\alpha Lz)\right] \tag{9.51}$$

$$
\begin{aligned}
&= C\left[\sin \alpha Lz + \frac{T_1}{2\alpha}(\cosh \alpha Lz - \cos \alpha Lz)\right] \\
&- C\left[\frac{T_1(\cosh \alpha L - \cos \alpha L) + 2\alpha L \sin \alpha L}{T_1(\cosh \alpha L - \cos \alpha L) + 2\alpha L \sinh \alpha L}\right]\left[\sinh \alpha Lz + \frac{T_1}{2\alpha}(\cosh \alpha Lz - \cos \alpha Lz)\right]
\end{aligned}
\tag{9.52}
$$

$$
= C\left\{\begin{array}{l} \sin \alpha Lz + \frac{T_1}{2\alpha}(\cosh \alpha Lz - \cos \alpha Lz) - \left[\frac{2\alpha L \sin \alpha L + T_1(\cosh \alpha L - \cos \alpha L)}{2\alpha L \sinh \alpha L + T_1(\cosh \alpha L - \cos \alpha L)}\right] - \\ \left[\sinh \alpha Lz + \frac{T_1}{2\alpha}(\cosh \alpha Lz - \cos \alpha Lz)\right] \end{array}\right\} \tag{9.53}
$$

$$
y(z) = C\left\{\begin{array}{l} \sin \alpha LZ \; - \left[\frac{2\alpha L \sin \alpha L + T_1(\cosh \alpha L - \cos \alpha L)}{2\alpha L \sinh \alpha L + T_1(\cosh \alpha L - \cos \alpha L)}\right] - \\ \left[\frac{T1(\sinh \alpha L - \sin \alpha L)}{2\alpha L \sinh \alpha L + T_1(\cosh \alpha L - \cos \alpha L)}\right](\cos \alpha Lz - \cosh \alpha Lz) \end{array}\right\} \tag{9.54}
$$

Or the above equation can also be written in form of 1/T1-

$$
= C\left\{\sin \alpha Lz - \left[\frac{(\cosh \alpha L - \cos \alpha L) + (1/T_1)\alpha L \sin \alpha L}{(\cosh \alpha L - \cos \alpha L) + (1/T_1)2\alpha L \sinh \alpha L}\right](\cos \alpha Lz - \cosh \alpha Lz)\right\} \tag{9.55}
$$

We can easily see that above equation reduces to the following for $1/T_1 \Rightarrow \infty$ i.e.,

$$y(z) = C\left\{\sinh \alpha L - \sin \alpha L - \frac{(\sinh \alpha L - \sin \alpha L)}{(\cosh \alpha L - \cos \alpha L)}(\cos \alpha zL - \cosh \alpha Lz)\right\} \tag{9.56}$$

By applying the boundary conditions we obtain the exact frequency equation (9.49) and the mode shape expression (9.55) and at T=$\propto$ reduces to equation (9.56)

We neglect the effect of internal pressure, P and rotatory inertia, J and consider the influence of rotational restraint parameter, T only on seismic response.

We know

$$c = \sqrt{\frac{P\pi R_m^2 J \omega^2}{EI}}$$

Since P and J are equal to zero, c=0 and if the roots of differential equation are α, β
If $\alpha=\beta$, then substituting c=0 in (9.16) we get-

$$\alpha = \sqrt{-c^2 + \sqrt{c^4 + \lambda^4}}$$

$$\therefore \alpha = \sqrt{\lambda^4} \tag{9.57}$$

9.5 Exact Analysis –Seismic Response

The various dimensional parameters of U-shaped bellows are taken to be same as those that have been considered by Morishita et al. [40]-

Bellows inner diameter (D_i) –545.8mm, convolution height, h=30mm; convolution pitch, p=25mm; bellows thickness, t_b=0.563mm: Bellows mean diameter, D_m=550.8mm: number of convolutions, N=20: pitch diameter of bellows, D_p=575.8mm and Bellows length L = 0.5 m respectively.

The total mass m_{tot} of bellows per unit length is found out using the expression given in chapter 3-

m_{tot} = 440.76kg/m.

From (9.4) we get -

$$\lambda = 7.0016\sqrt{\omega} \tag{9.58}$$

The rotational restraint parameters T_1 and T_2 are varied from a minimum value of 0.01 to a maximum of 10^{10}. However, it is assumed that there are equal rotations on either ends i.e.$T=T_1=T_2$.

The maximum response displacement of any arbitrary section of bellow to a given response spectrum is given by-

$$y_{max} = \beta \frac{S_a}{(2\pi f)^2} \cdot \frac{D_p}{2.N.p} y(z) \tag{9.59}$$

Where β is participation factor and is assumed to be equal to 1.0

y(z) is maximum deflection at the central convoluted portion of the bellows and reduces gradually from crown to root. Therefore, y(z) is taken as 1.0.

S_a is the spectral gravitational acceleration at frequency 'f' and has units of mm/s^2.

The bending stress induced by lateral vibration of bellows is not caused by lateral displacement of crowns directly, but by axial displacement between adjacent crowns, which is the deflection of neutral axis of bellows. So we must relate the axial and lateral displacements in beam mode vibration of bellows.

9.6 Results and Discussion

Table 9.1 gives a comparison of the exact frequencies obtained using the bisection method for a single bellows that are elastically restrained at both ends vis-à-vis to the fixed-fixed end results presented by Morishita et al. [5].

Table 9.1 Comparison of Frequencies at T= ∝

Mode #	Morishita [5] f, Hz	Exact f, Hz
1	8.6	8.05

It is seen that the frequencies obtained by considering the ends as elastically restrained are lower than that was found out by Morishita et al. Therefore, it can be concluded that the seismic response by exact method is accurate.

Table 9.2 presents the first mode frequency at $T=T_1=T_2$. The time period, t in sec and the spectral acceleration, S_a values are obtained.

Table 9.2 Fundamental Frequency for various values of restraint parameter T=0.01 to ∝

T	f (Hz)	t (sec)	Sa (mm/s^2)
0.01	0.8023	1.25	79
0.1	0.8061	1.24	105
1.0	1.6826	0.6	473
10	1.7134	0.58	479
10^2	1.8101	0.553	500
10^3	1.8114	0.55	498
10^4	2.6169	0.382	763
10^5	3.4224	0.292	1513
10^6	4.2279	0.237	1315
10^7	5.0328	0.12	2000
10^8	5.6366	0.177	2210
10^9	6.4427	0.155	3000
10^{10}	7.2476	0.138	2658
10^{11}	8.0525	0.124	2263

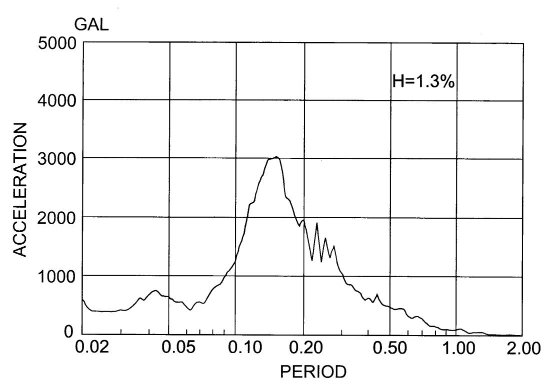


Fig 9.2 Time History Curve of Seismic Response

It is observed that for N=1 and at T=0.01 the time period found out is 0.49sec, while at T=∝, the time period is 0.12sec. It is seen that as rotational stiffness increases and approaches infinity, the time period of vibration decreases. A standard

time history graph of seismic wave and its acceleration response is used for obtaining the seismic accelerations (S_a) at various frequencies as shown in Fig 9.1.

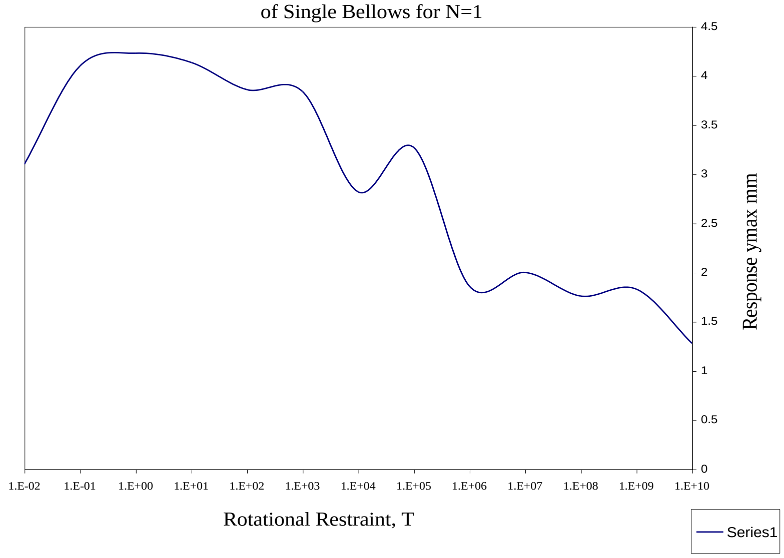
Table 9.3 gives the maximum displacement, y_{max} and bending stress σ_b values for varying rotational restraint parameter T.

Table 9.3: Response of Bellows to Lateral Seismic excitation

T	y_{max}(mm)	σ_b(kg/mm^2)
0.01	3.1	21.8
0.1	4.1	28.9
1.0	4.2	29.7
10	4.1	28.8
10^2	3.86	27.1
10^3	3.84	27.0
10^4	2.82	19.8
10^5	3.27	23.0
10^6	1.86	13.1
10^7	2.0	14.0
10^8	1.76	12.4
10^9	1.83	12.9
10^{10}	1.28	9.0
10^{11}	0.88	6.22

The seismic gravitational acceleration obtained at $T=\propto$ and time period of t=0.124sec is 2263mm/s^2. Substituting this value in equation (9.59), the maximum lateral displacement is obtained as y_{max} = 0.88mm compared to 0.84mm by Morishita et al [5] at $T=\propto$.

Fig 9.3: Influence of Rotational Restraint on Seismic Response of Single Bellows for N=1



The exact method developed is to calculate the dynamic characteristics and seismic response of bellows for its lateral vibrations. The theoretical formulations are based on the equations of an equivalent Euler-Bernoulli beam, and the bending stress due to these vibrations is estimated. In case of lateral vibrations it is found that the influence of elastically rotational restraint at either ends to its natural frequencies is significant.

Also, it can be seen from fig 9.2 that there is a considerable influence of rotational restraint on seismic response. As the rotational restraint parameter T increases from 0.1 to 1.0, there is a steep rise in seismic response followed by a fall at $T=10^2$. Beyond $T=10^2$, the seismic response falls up to $T=\propto$, except at 10^6 where there is a rise because of sudden peak in acceleration.

CHAPTER 10

CONCLUSIONS

10.1 Conclusions

It is important to know the natural frequencies for predicting the dynamic response of structures. The system under consideration in the thesis is the bellows expansion joints. Several types of expansion joints are used in practice. However, the thesis considers two types –single and double or Universal type of expansion joints only. Theoretical models for the investigation of axial and transverse vibrations in single and double bellows were developed in the thesis. The Euler-Bernoulli and Timoshenko Beam differential equations were used to derive the "exact frequency equations" with necessary boundary conditions. It is seen that other researchers had considered classical fixed-fixed end conditions only in determining the natural frequencies.

However, in practice it is observed on many occasions that the pipes are subjected to equal/unequal rotations. Hence, the effects of variation of the rotational restraint parameter, internal pressure and flow velocity on natural frequencies were studied by considering elastically restrained ends.

The exact solutions were then compared quantitatively with approximate solutions of other investigators to verify their accuracy. The finite element method

was used in establishing the closeness of values with exact solutions and also experimental investigations.

In order to simplify the double bellows expansion joint problem- it was divided in to two separate problems for lateral and rocking modes respectively. These two problems are governed by the same differential equation but with two different boundary condition sets.

Comparison of these calculated natural frequencies with experiments conducted by other researchers showed very close agreement. This can be explained by the simplistic classical approach considered by others and the approximate methods used in finding out the natural frequencies.

The analysis provided in the thesis demonstrated that by considering the bellows as a “beam”, the shear influence is negligible because of the transverse flexibility of bellows. On the other hand the rotatory inertia of bellows cross-section including the fluid trapped is an important factor for the natural frequencies of bellows, as does the internal pressure. Similarly, the effect of the variation of support mass and support stiffness on natural frequencies is studied in case of Universal type of expansion joint.

The exact solutions were within 10% of error with results of finite element analysis and other investigators.

The dimensions of bellows chosen in the thesis are same as those considered by Jakubauskas.V.F, Li Ting-Xin and Morishita et al, -only to arrive at logical comparison purpose. The bellows considered here were of single ply construction.

10.2 Concluding Remarks

The investigations reported in literature demonstrate that the expressions currently used for the computation of axial and transverse frequencies in bellows are not adequate for reasonable predictions in design practice. It was found that the elastically restrained end conditions in the presence of internal pressure and flow velocity was ignored by previous authors.

10.3 Future Scope of Studies

The criteria for vibrations of multi-ply bellows goes beyond the scope of this thesis. Hence, for establishing the same and for better understanding of axial and transverse vibrations of single and double multi-ply bellows –further studies are recommended. Research on composite material bellows is also interesting and worth attempting.

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